Practicing Proofs
MAT246 Handouts

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Chapter 1

Introduction

1.1 A note to students: practice in the gen AI era

Learning to write proofs is a doable and learnable process that takes practice and reflection. These handouts are intended as practice, not texts you read like a novel or newspaper article. In order to get the most out of this collection of practice problems, you should try the problems yourself, get stuck, and let yourself be stuck for a while before getting a hint or reading a solution. The reason why you should do this is because the process of getting stuck and then unstuck is one of the best ways to support your brain's development and growth.

Learning how to prove theorems and understand the mathematical process is a developmental process. It's like learning a musical instrument or training for a marathon. You have to build your mind and body to do these activities. Learning math is like that. Getting stuck, thinking about your thinking (i.e. metacognition), and giving yourself enough time figure things out is how you get smarter. This is sometimes called practice.

If you depend too much on watching videos or reading someone else's solutions, it's like watching someone else play an instrument or train for a marathon. It might help on some basic level, but your mind and your body are not getting smarter and stronger because you are not putting in the time.

Being smart is not a fixed ability or attribute like the colour of your eyes. In fact, believing that you can get smarter and knowing that the ingredients of being smart is to practice with intent is how you can continue to grow, and hence "be smart."

One important aspect of learning mathematical thinking and proof is that a large part of it is developing your thinking rather than gathering facts. It's learning ways of thinking that are new to you, and that takes practice. Hence, getting answers using an "information gathering lens" is not sufficient and likely limits your growth.

Suggested processes that you can adapt.

1. Try a problem
2. If you got it, great! Reflect (see below!), try it another way, and move on.
3. If stuck, take your time and embrace being stuck. Try to get "unstuck" by thinking about the material you've been learning, recalling similar problems, trying to understanding the problem better using small examples, and so on. This is where learning actually occurs!
4. After you feel like you've been spinning your wheels long enough (say 20–30 minutes as a loose approximation), look at the hints section; start with the first hint but do not read further! Start the process again from the beginning and go back for another hint if you're still genuinely stuck.

5. Repeat until you have a solution you believe in.

6. Review your proof and the solution and reflect on what the big ideas are, where you can improve, and so on.

7. The point of these problem is not "to get through them" (there isn't a medal at the end!). Rather, they are a tool for growing your mathematical ability. To get the most out of the problems you should reflect on each one after you're done: are there other ways of solving it? is it similar to previous problems? why did the text suggest this problem? is there a way to make the solution clearer?

(Practice in the gen AI era) Having a healthy attitude and disposition towards practice in the "gen AI" era has extra importance. We all know as of this writing in 2025 that nearly everyone uses generative AI or has access to it. There are emerging studies that show lower brain use, brain development, and learning outcomes for students who use generative AI instead of using their brains. It benefits your growth to know that *It's is okay to be stuck* and *It's better for you to not use AI*. In fact, it's better for you that you allow yourself to get stuck for a while, because it is the process of being stuck and getting unstuck where you learn the most. When you use generative AI too soon and too often, you are taking away opportunities train your mind. This is like not spending enough time practicing music or not putting in enough miles to train for a marathon. While it is tempting to save time to get that assignment done, ultimately in the long run you will grow less and be less prepared to solve the real problems you will face in the future, long after your university years. There is no substitute for practice time and volume of effort. Time, effort, dedication, and practice are how you get better and grow smarter.

1.2 Some details

These handouts are linked to the textbook by Professor Dana Ernst, Northern Arizona University. It is suggested that you download the book as use it as a reference. You can find the textbook here for free https://danaernst.com/IBL-IntroToProof/. Hard copies can be ordered from the AMS/MAA Press. Proceeds from the textbook are donated to Association for Women in Mathematics.

These handouts are intended to support students in MAT246 at the University of Toronto. They can be used and benefit students no matter what textbook is used. The handouts are also useful for students in other courses and contexts, where mathematical thinking and proof are needed.

Please send feedback to math.undergrad@utoronto.ca with subject "MAT246 Practice Materials Feedback."

1.3 A note to instructors

These handouts can be a useful resource for an Intro to Proofs course, such as MAT246. They are intended to be used with Ernst's textbook. However, the handouts are self-contained and can supplement any other textbook.

Sample lecture handouts that can be used with or without active lecture and pair/group work are also available for a handful of sections. These are based on the Ernst textbook and provide a framework for how to organize your class time.

Research in undergraduate Mathematics Education suggests strongly that students benefit significantly from tasks that engage students in sense making. For example using think-pair-share and its variations, where students are asked regularly and consistently to think about an idea and discuss with a partner can support student learning outcomes and surface learning needs that you can address on the spot. Mixing in active learning in your lectures has significant benefit and does not take significant prep time.

Sections of the Ernst textbook (Spring 2025 version) covered by these handouts are listed below.

- 2.1, 2.2, 2.3, 2.4, 2.5
- 3.1, 3.3, 3.4, 3.5
- 4.1, 4.2, 4.3, 4.4
- 7.1, 7.2, 7.3, 7.4
- 8.1, 8.2, 8.3, 8.4

Sample lecture schedule. Time needs in your class may vary.

- 2.1 (2 hours)
- 2.2 (2 hours)
- 2.3 (2 hours)
- 2.4 (1 hour)
- 2.5 (2 hours)
- 3.1 (2 hours)
- 3.2, 3.3 (Cover briefly)
- 3.4 (1 to 2 hours)
- 3.5 (1 hour)
- 4.1 (2 hours)
- 4.2 (1 hour)
- 4.3 (1 hour)
- 4.4 (1 hour)
- 7.1 (1-2 hours)
- 7.2 (2 hours)
- 7.3 (1 hour)
- 7.4 (1 hour)
- 8.1 (1 hour)
- 8.2 (2 hours)
- 8.3 (1 hour)
- 8.4 (1 hour)

Part I

Practice

Chapter 2

Definitions

Exercise 1 (Vocabulary)**.** *This exercise will help you practice the mathematical symbols, vocabulary, and syntax (grammar) that was introduced in the text.*

The following symbols were introduced in the text:

$$:= \qquad \in \qquad | \qquad \mathbb{N} \qquad \mathbb{Z} \qquad \mathbb{R}$$

For each symbol above:

(a) explain its meaning; compare and contrast it with other symbols or closely-related concepts.

(b) give an example of correct usage and the meaning of your example;

(c) give an example of an incorrect usage and explain the error.

(Hints on page 169. Solutions on page 323.)

Exercise 2 (Parity)**.** *The goal of this exercise is to help you practice* mathematical *definitions and how to apply them.*

Here is the definition of "even" and "odd" from the text:

> **Definition 2.1.** An integer n is **even** if $n = 2k$ for some $k \in \mathbb{Z}$. An integer n is **odd** if $n = 2k + 1$ for some $k \in Z$.

(a) Use the definition above to *prove* that 101 is odd and 246 is even.

(b) Can we use the definition to *prove* that 101 is *not* even? If so, how? If not, what else would we need?

(Hints on page 170. Solutions on page 325.)

Exercise 3 (Squaring)**.** *The goal of this exercise is to help you practice* mathematical *definitions and how to use them to prove novel results.*

Use Definition 2.1 above to prove the following theorem from the text:

> **Theorem 2.2.** If n is an even integer, then n^2 is an even integer.

(a) Where in the proof have you used the definition?

(b) Apart from using Definition 2.1, did you make any *assumptions* in your proof? Can you identify them and clearly state each one?

(Hints on page 171. Solutions on page 326.)

Exercise 4 (Divisibility)**.** *This exercise shows how new definitions generalize previous definitions. Claims of "generalization" require proofs which formalize how two formally defined concepts relate to each other.*

Below is the definition of divisibility from the text:

> **Definition 2.5.** Given $n, m \in \mathbb{Z}$, we say that n **divides** m, written $n|m$, if there exists $k \in \mathbb{Z}$ such that $m = nk$. If $n|m$, we may also say that m is **divisible by** n or that n is a **factor** of m.

(a) Use Definition 2.1 and Definition 2.5 to prove that if n is an even integer, then it is divisible by 2.

(b) Use Definition 2.1 and Definition 2.5 to prove that if n is divisible by 2 then it is an even integer.

(c) What is the difference between the statements in parts (a) and (b) of this question? Do they mean the same thing? Are the proofs the same? Can we use one of the statements to prove the other?

(d) Use Definition 2.5 to state precisely what it means to say that n is *not* divisible by 2.

(e) Suppose n is an odd integer; can we conclude that n is not divisible by 2? Why or why not?

(Hints on page 172. Solutions on page 327.)

Chapter 3

Propositional Logic

Exercise 5 (Translation)**.** *In this exercise you will practice expressing logical ideas in mathematical language using the symbols of propositional logic.*

Consider the following propositions:

$$R = \text{It is currently raining in Toronto;}$$
$$U = \text{Mai the Mathematician is holding an umbrella;}$$

Use the symbols R, U (representing the above propositions) and the connective symbols $\neg, \vee, \wedge, \implies, \iff$ to express each of the following compound propositions symbolically:

(a) If it is currently raining in Toronto, then Mai the Mathematician is holding an umbrella.

(b) It is not currently raining in Toronto.

(c) It is currently raining in Toronto or Mai the Mathematician is holding an umbrella.

(d) Mai the Mathematician is holding an umbrella if and only if it is currently raining in Toronto.

(e) It is currently raining in Toronto and Mai the Mathematician is holding an umbrella.

(f) Whenever Mai the Mathematician is not holding in umbrella, it is not raining in Toronto.

(Hints on page 173. Solutions on page 328.)

Exercise 6 (Truth Values). *In this exercise you will practice assigning truth-values to compound propositions that are written using logical connectives.*

For each compound propositions from your answer to Exercise 5, describe a situation in which the proposition is true and a situation in which the proposition is false. (By “situation” we mean the weather conditions and umbrella-holding.)

(Hints on page 174. Solutions on page 329.)

Exercise 7 (Truth Tables)**.** *In propositional logic, connectives are defined by their action on truth values. This exercise will help you practice these definitions.*

A *proposition* is a statement that must have exactly one of the values "true" or "false". It is sometimes customary to represent the value "false" by 0 and the value "true" by 1. Recall that propositions can be combined into compound propositions using *connectives*. Complete the *truth-tables* for each of the following connectives.

A	$\neg A$
0	
1	

A	B	$A \wedge B$
0	0	
0	1	
1	0	
1	1	

A	B	$A \vee B$
0	0	
0	1	
1	0	
1	1	

A	B	$A \implies B$
0	0	
0	1	
1	0	
1	1	

A	B	$A \iff B$
0	0	
0	1	
1	0	
1	1	

A	B	$\neg A$	$(\neg A) \vee B$	$A \implies B$
0	0			
0	1			
1	0			
1	1			

A	B	$A \implies B$	$\neg(A \implies B)$	$\neg B$	$A \wedge (\neg B)$
0	0				
0	1				
1	0				
1	1				

A	B	$A \implies B$	$B \implies A$	$(A \implies B) \wedge (B \implies A)$	$A \iff B$
0	0				
0	1				
1	0				
1	1				

(Hints on page 175. Solutions on page 331.)

Exercise 8 (Arithmetic)**.** *Propositional logic is the basis for almost all digital technology. The truth values are expressed as binary* 0 *and* 1 *or "on" and "off". The logical connectives (or* logic gates*) are functions of these states. This exercise explores some of these functions.*

Let us revisit the truth-tables from your answer to Exercise 7, where you have used 0 and 1 to represent truth values. Can you represent the connectives $\neg, \wedge, \vee, \implies, \iff$ as arithmetic functions using addition, multiplication, subtraction, and constants? For example, for any proposition A with truth value either 0 or 1, we have can express $\neg A$ as the arithmetic function $1 - A$ since

A	$\neg A$	$1 - A$
0	1	1
1	0	0

(a) Study the truth-table of $\wedge$. Can you express $A \wedge B$ as a familiar arithmetic function of A and B?

(b) Study the truth-table of $\vee$. Can you express $A \vee B$ as a (combination of) familiar arithmetic functions of A and B?

(c) Study the truth-table of $\implies$. Can you express $A \implies B$ as a (combination of) familiar arithmetic functions of A and B?

(d) Study the truth-table of $\iff$. Can you express $A \iff B$ as a (combination of) familiar arithmetic functions of A and B?

This is essentially how computers represent logical operations!

(Hints on page 176. Solutions on page 332.)

Exercise 9 (Propositions)**.** *In this exercise you will practice translating symbolic language to a more intuitive mathematical language. This skill is of significant use in learning novel mathematical ideas.*

Consider the following propositions:

$E =$ "The integer 2 is an even number";
$F =$ "The integer 4 is an even number";
$P =$ "The integer 2 is a prime number";
$Q =$ "The integer 4 is an even number".

For each compound proposition below: write it in mathematical English and determine whether it is true or false.

(a) $E \wedge P$

(b) $E \vee P$

(c) $E \implies (\neg E)$

(d) $(\neg E) \implies E$

(e) $P \implies E$

(f) $E \iff P$

(g) $(\neg E) \iff P$

(h) $F \iff Q$

(Hints on page 177. Solutions on page 333.)

Exercise 10 (Cards)**.** *This classic exercise asks you to reflect on the meaning of mathematical implication (also called "material implication").*

An unusual deck of alphanumeric cards consists of cards each of which has one of the letters A through Z on one side and one of the numerals 0 through 9 on the other side. Four cards are arranged on the table

$$1 \qquad\qquad 2 \qquad\qquad A \qquad\qquad B$$

Your mathematician friend advances the following hypothesis about the cards on the table:

> If a card has an even number on one side, then it has a vowel on the other wide.

(a) Translate this hypothesis to a compound proposition using appropriate connectives. Remember to clearly define your component propositions.

(b) Which cards (if any) must be turned over in order to test (verify or falsify) this hypothesis?

(Hints on page 178. Solutions on page 334.)

Exercise 11 (Nested)**.** *This exercise continues the reflection on the meaning of material implication, but this time via its negation.*

Let $A, B, C, \ldots, Z$ be propositions.

(a) Suppose it is know that the proposition $A \implies B$ is *false*; what is the truth-values of A and B in this case?

(b) Suppose it is known that the proposition $B \implies A$ is *false*; what is the truth-values of A and B in this case?

(c) Suppose it is known that the proposition $A \implies (B \implies C)$ is *false*; what is the truth-value of A and B in this case?

(d) Suppose it is known that the proposition $(A \implies B) \implies C$ is *false*; what is the truth-value of A, B, and C in this case?

(e) Suppose it is known that the proposition below is *false*:

$$A \implies (B \implies (C \implies (\cdots (Y \implies Z) \cdots))).$$

What is the truth-value of $A, B, \ldots, Z$ in this case?

(Hints on page 179. Solutions on page 335.)

Exercise 12 (Words). *Mathematical implication is fundamental to stating theorems and other mathematical results; these results are often stated in writing instead of symbols. In this exercise you will practice deciphering the meaning of common phrases that are connected to the implication symbol.*

Consider the compound proposition $A \implies B$.

(a) Use logical connectives to express each of the *converse*, the *inverse*, and the *contrapositive*. Which ones (if any) are logically equivalent to the implication $A \implies B$?

(b) Use the implication symbol "$\implies$" to express each of the following word-descriptions:

(i) A is necessary for B.

(ii) A is sufficient for B.

(iii) A only if B.

(iv) A if B.

(v) A whenever B.

(Hints on page 180. Solutions on page 336.)

Exercise 13 (Equivalence)**.** *While writing proofs mathematicians will switch between logically equivalent statements without warning. This exercise will help you practice identifying and switching between logically equivalent statements.*

Let A and B be propositions. Which of the following compound propositions are logically equivalent to each other? Collect them into groups.

(a) $A \implies B$

(b) $B \implies A$

(c) $(\neg A) \implies (\neg B)$

(d) $(\neg B) \implies (\neg A)$

(e) $\neg(A \implies B)$

(f) $(\neg A) \wedge (\neg B)$

(g) $(\neg A) \vee (\neg B)$

(h) $(\neg A) \vee B$

(i) $A \vee (\neg B)$

(j) $(\neg A) \wedge B$

(k) $A \wedge (\neg B)$

(l) $\neg(A \wedge B)$

(m) $\neg(A \vee B)$

(Hints on page 181. Solutions on page 337.)

Exercise 14 (Complete Set of Connectives)**.** *Logic gates (see Exercise 8) are often constructed from a few simple types in order to economize on production. This exercise explores which connectives can be expressed in terms of which other connectives.*

Let A and B be proposition. Consider the five compound propositions introduced in the texts:

$$\neg A \qquad A \wedge B \qquad A \vee B \qquad A \implies B \qquad A \iff B.$$

(a) For each compound proposition above, find a logically equivalent proposition which only uses the two connectives $\neg, \vee$; these two connectives are said to form a *complete set of connectives*.

(b) For each compound proposition above, find a logically equivalent proposition which only uses the two connectives $\neg, \wedge$; these two connectives also form a complete set of connectives.

(c) For each compound proposition above, find a logically equivalent proposition which only uses the two connectives $\neg, \implies$; these two connectives are another example of a complete set of connectives.

(d) Consider a new connective, $\uparrow$ with the following truth table

A	B	$A \uparrow B$
0	0	1
0	1	1
1	0	1
1	1	0

For each compound proposition above, find a logically equivalent proposition which only uses the connective $\uparrow$. This is an example of a single connective which (by itself) forms a complete set of connectives.

(Hints on page 182. Solutions on page 338.)

Exercise 15 (Tautologies and Contradictions)**.** *In this exercise we will practice identifying tautologies and contradiction. It will also help you practice logical equivalence, as well as working and interpreting the logical connectives (especially mathematical implication).*

Let A, B, and C be propositions. For each of the following compound propositions determine (with justification) whether it is a tautology, a contradiction, or neither.

(a) $((A \wedge B) \implies C) \implies (A \implies (B \implies C))$.

(b) $((\neg A) \wedge B) \implies ((\neg B) \vee C)$.

(c) $(A \implies (B \implies C)) \wedge ((A \wedge B \wedge (\neg C))$.

(Hints on page 183. Solutions on page 339.)

Chapter 4

Proving Conditional Propositions

Exercise 16 (Contrapositive Statements)**.** *In this exercise you will practice translating between an implication and its logically equivalent contrapositive form.*

Consider the following list of implications. Find the contrapositive for each of them.

(a) If $x > 3$ then $x + 2 > 5$.

(b) If today is Wednesday, then tomorrow is Thursday.

(c) If f is differentiable at x, then f is continuous at x.

(d) If n is a multiple of 6, then n is a multiple of 3.

What is the truth value of these statements? What about their contrapositive?

(Hints on page 184. Solutions on page 340.)

Exercise 17 (Direct Proofs)**.** *This exercise will help you practice writing direct proofs and the useful technique of "unpacking" mathematical statements.*

Give a direct proof of the following conditional statements.

(a) If a and b are integers and $a \mid b$, then $a \mid bc$ for every integer c.

(b) If m and n are both odd integers, then $m + n$ is even.

(c) If $a \mid b$ and $b \mid c$, then $a \mid c$.

(d) If n is divisible by 6, then n is divisible by 2 and 3.

(Hints on page 185. Solutions on page 341.)

Exercise 18 (Contra- Proofs). *Often it is extremely challenging to prove an implication directly, but relatively straightforward to prove it by contradiction. Proofs by contradiction "add assumptions" by negating the conclusion. Similarly, for many statements it is much easier to prove the contrapositive, which "change assumptions" by replacing the hypothesis with the negation of the conclusion.*

Prove each of the following conditional statements by contrapositive *and* by contradiction. Throughout this exercise m, n are assumed to be integers.

(a) If n^2 is even, then n is even.

(b) If n^2 is not divisible by 3, then n is not divisible by 3.

(c) If $7n^3$ is odd, then n is odd.

(d) If $m \cdot n$ is even, then at least one of m, n is even.

(Hints on page 186. Solutions on page 342.)

Exercise 19 (Direct Proofs II). *In this exercise we continue to practice direct proofs; it has the added benefit of practicing manipulating inequalities, an important skill for many proofs and applications!*

Give a direct proof of the following conditional statements; you may use familiar facts about real numbers and inequalities from high-school algebra.

(a) If $x > 3$, then $x^2 > 9$.

(b) If $0 < x < y$, then $x^2 < y^2$.

(c) If $0 \leq x \leq 1$, then $x^2 \leq x$.

(d) If $0 < x < y$, then $\frac{1}{x} > \frac{1}{y}$.

(e) If $|x| < 1$, then $x^2 < 1$.

(Hints on page 187. Solutions on page 343.)

Exercise 20 (Contra- Proofs II)**.** *In this exercise we continue to practice proofs by contrapositive and contradiction; an added benefit is manipulating (the negation of) inequalities.*

Prove the following conditional statements by contrapositive *and* by contradiction.

(a) If $a^2 \neq b^2$, then $a \neq b$.

(b) If $x \geq 0$, then $x \leq 1 + x^2$.

(c) If $|x| > 5$, then $x^2 > 25$.

(d) If $|x - 3| < 2$, then $1 < x < 5$.

(e) Use a proof by contradiction to show that if $x^2 = 2$, then x is not rational.

(Hints on page 188. Solutions on page 344.)

Exercise 21 (Practicing Proofs)**.** *This exercise is a more challenging "capstone exercise". You may use any of the techniques we've learned.*

Prove each of the following statements.

(a) If m and n are integers and $m^2 + n^2$ is even, then m and n are both even or both odd.

(b) If n is an integer, then $n^2 - n$ is even.

(c) If m and n are integers and $m + n$ is odd, then $m^2 + n^2$ is odd.

(d) If m and n are integers, then mn is even if and only if m is even or n is even.

(e) If n is odd, then $8 \mid (n^2 - 1)$.

(f) If $x \geq 0$ and $y \geq 0$ are nonnegative real numbers, then $x + y \geq 2\sqrt{xy}$.

(Hints on page 189. Solutions on page 345.)

Chapter 5

Quantifiers

Exercise 22 (Proposition vs. Predicate). *This exercise will help you practice the definitions of proposition and predicates.*

Below are several logical expressions. For each one:

(i) Determine whether it is a **proposition** (i.e., has a definite truth value). If it is a proposition, what is its truth value? If it is not a proposition, explain why (e.g., it has a free variable).

(ii) Determine if the statement is a **predicate** (a formula that becomes a proposition when variables are bound). State which variables, if any, are free, and which variables are bound.

(a) The sun is hot

(b) Where is Waldo?

(c) $x^2 > 4$

(d) $P(y)$, where $P(y) := y < 1$

(e) $Q(0)$, where $Q(x) := x > 1$

(f) There exists some integer z such that $2z + 1 = 1$

(g) For every real number r, $r > 1$

(Hints on page 190. Solutions on page 347.)

Exercise 23 (Vocabulary of Quantifiers)**.** *This exercise will help you practice the definition and usage of the most common logical quantifiers.*

One of your classmates has missed the lecture and asks you to explain the logical quantifiers $\forall$ and $\exists$. Please help them understand how to use these symbols!

In addition to explaining their meaning, make sure to provide some example of correct mathematical usage and also incorrect grammatical usage.

(Hints on page 191. Solutions on page 348.)

Exercise 24 (Finite Universe of Discourse)**.** *This exercise will further your understanding of the logical quantifiers by showing how their meaning can be "unpacked" when dealing with finite universes.*

Suppose the universe of discourse for a predicate $P(x)$ is the set $\{1, 2, 3, 4, 5\}$. Express the following statements without using quantifiers. Use formal logical notation including the predicate P, negations, conjunctions and disjunctions.

(a) $\forall x\, P(x)$

(b) $\exists x\, P(x)$

(c) $\neg(\exists x\, P(x))$

(d) $\neg(\forall x\, P(x))$

(e) $[\forall x\, ((x \neq 3) \implies P(x))] \vee [\exists x\, (\neg P(x))]$

Bonus: Suppose $P(x) := x > 0$; interpret the meanings of the statements above in plain English and determine their truth values.

(Hints on page 192. Solutions on page 349.)

Exercise 25 (Changing the Universe of Discourse)**.** *The purpose of this exercise is to illustrate that the meaning (and truth value) of a quantified statement depends on the universe of discourse.*

For each proposition below, evaluate its truth values when the universe of discourse is $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{R}$.

(a) $\forall x\,(x^2 \geq 0)$.

(b) $\forall x\,(x > -1)$.

(c) $\exists x\,(x > 0 \wedge x < 1)$.

(d) $\exists x\,(x + 1 < 0)$.

(e) $\forall x\,((x \neq 0) \implies x \text{ is not a solution to } x^2 = 2)$.

(Hints on page 193. Solutions on page 351.)

Exercise 26 (Translating Quantified Statements)**.** *This exercise will help you practice statements with more than one variable, where one needs pay attention to both the type and order of quantification.*

Suppose the universe of discourse is all students at UofT, and consider the predicate $K(x, y) :=$ "x knows y". Interpret the meaning of each of the following statements. Pay attention to how changing the quantifiers and their order changes the meaning of the statement.

(a) $\forall x \, \forall y \, K(x, y)$.

(b) $\forall x, y \, (K(x, y) \implies K(y, x))$.

(c) $\forall x \, \exists y \, K(x, y)$.

(d) $\exists x \, \forall y \, K(x, y)$.

(e) $\exists x \, \exists y \, K(x, y)$.

(f) $\exists x, y \, ((x \neq y) \wedge K(x, y))$.

(Hints on page 194. Solutions on page 352.)

Exercise 27 (Evaluating Quantified Statements). *This is a "capstone exercise" which will help you evaluate your comfort with interpreting quantified statements.*

For each statement below decide whether it is true or false, and whether it remains so if the order of quantifiers is reversed. Explain your reasoning.

(a) $\forall x \in \mathbb{R}\, \exists y \in \mathbb{R}\, (x + y = 0)$.

(b) $\forall x \in \mathbb{N}\, \exists y \in \mathbb{N}\, (x < y)$.

(c) $\forall x \in \mathbb{Z}\, \exists y \in \mathbb{Z}\, (x + y = 7)$.

(d) $\forall x \in \mathbb{R}\, \exists y \in \mathbb{R}\, (y^2 = x)$.

(e) $\forall x \in \mathbb{R}\, \exists y \in \mathbb{R}\, (y = x^2)$.

(Hints on page 195. Solutions on page 353.)

Exercise 28 (Exchanging Quantifiers)**.** *This exercise will help you reflect on the order of quantification.* Suppose U is some universe of discourse and $P(x, y)$ some predicate in the variables x, y.

(a) Is it possible that $\forall x\, \exists y\, P(x, y)$ is *true* but $\exists x\, \forall y\, P(x, y)$ is *false*? Prove your answer.

(b) Is it possible that $\forall x\, \exists y\, P(x, y)$ is *true* but $\exists y\, \forall x\, P(x, y)$ is *false*? Prove your answer.

(c) Is it possible that $\exists x\, \forall y\, P(x, y)$ is *true* but $\forall x\, \exists y\, P(x, y)$ is *false*? Prove your answer.

(d) Is it possible that $\exists y\, \forall x\, P(x, y)$ is *true* but $\forall x\, \exists y\, P(x, y)$ is *false*? Prove your answer.

(Hints on page 196. Solutions on page 354.)

Chapter 6

Introduction to Sets

Exercise 29 (Set-Builder Notation)**.** *In this exercise, you'll practice packing and unpacking sets using set-builder notation. Sometimes it's easier to describe a set with a rule (packing), and other times it's clearer to list out its elements (unpacking).*

(a) List the elements of the set $\{n \in \mathbb{N} \mid n < 5\}$.

(b) List the elements of the set $\{x \in \mathbb{Z} \mid -2 < x \leq 2\}$.

(c) Express the set $\{\ldots, -4, -2, 0, 2, 4, \ldots\}$ in set-builder notation.

(d) Express the interval $(2, 5]$ in set-builder notation.

(Hints on page 197. Solutions on page 355.)

Exercise 30 (Subsets)**.** *The focus of this exercise is the definition of a subset and how to determine if one set is a subset of another.*

(a) For each pair, decide whether $A_i \subseteq B_i$:

$$A_1 = \{1, 2, 3\}, \qquad B_1 = \{1, 2, 3, 4\},$$
$$A_2 = \{1, 3, 5\}, \qquad B_2 = \{2, 4, 6\},$$
$$A_3 = \{\{1\}\}, \qquad B_3 = \{1, \{1\}\}.$$

(b) Give an example of sets A and B with $A \subsetneq B$.

(c) Show that $\emptyset \subseteq A$ for every set A.

(Hints on page 198. Solutions on page 356.)

Exercise 31 (Set Equality)**.** *At first glance, this problem might seem trivial once you recognize the elements on both sides. However, it is a useful exercise in reasoning formally: instead of listing elements, you will prove equality using the definition of set equality. Practicing this kind of structured argument builds habits that apply to more abstract proofs later on.*

Use the definition of set equality via double subset inclusion to prove that

$$\{1, 2\} = \{x \in \mathbb{R} \mid x^2 - 3x + 2 = 0\}.$$

(Hints on page 199. Solutions on page 357.)

Exercise 32 (Set Operations)**.** *In this exercise, you'll practice computing unions, intersections, differences, and complements of sets. Use the definitions to work out each result.*

Let $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$ be sets in the universe $U = \{1, 2, 3, 4, 5\}$. Compute the following sets (you do not need to prove anything, simply state your answer):

(a) $A \cup B$

(b) $A \cap B$

(c) $A \setminus B$

(d) $B \setminus A$

(e) A^c

(f) $(A^c)^c$

(g) B^c

(h) $(A \cup B)^c$

(i) $A^c \cap B^c$

(j) $(A \cap B)^c$

(k) $A^c \cup B^c$

Determine if the sets A and B are disjoint. Are the sets A and $B \setminus A$ disjoint?

(Hints on page 200. Solutions on page 358.)

Exercise 33 (The Empty Set). *In this exercise, you'll see how the empty sets interacts with unions, intersections, and complements.*

Prove that for set A in the universe U we have

(a) $A \cup \emptyset = A$

(b) $A \cap \emptyset = \emptyset$

(c) $\emptyset^c = U$.

(Hints on page 201. Solutions on page 359.)

Exercise 34 (Properties of Set Operations)**.** *In this exercise, you'll explore important properties of set operations. Some problems ask you to verify given statements by providing an example (note that this does not prove the statement, and you are welcome to try to prove it), while others require you to write full proofs using the definitions.*

(a) Prove Theorem 3.10 (Transitivity of Subsets). Prove that if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

(b) Verify Theorem 3.22 (Distribution of Union and Intersection) with a small example. In other words, give a concrete example of sets A, B and C and show that the following set equalities are satisfied:

  (i) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, and

  (ii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

(c) Show that the union operation is associative, more precisely, that $A \cup (B \cup C) = (A \cup B) \cup C$.

(d) Show that the intersection operation is associative, more precisely, that $A \cap (B \cap C) = (A \cap B) \cap C$.

(e) Show that for any set A in a universe U, we have $(A^c)^c = A$.

(Hints on page 202. Solutions on page 360.)

Exercise 35 (Subset Equivalences)**.** *This problem asks you to connect the definition of a subset with two equivalent conditions involving unions and intersections.*

Show that $A \subseteq B$ if and only if $A \cup B = B$, and that $A \subseteq B$ if and only if $A \cap B = A$.

(Hints on page 203. Solutions on page 362.)

Exercise 36 (Set Equalities)**.** *In this exercise, you'll prove several set equalities that involve complements. Use definitions and logical reasoning to show that both sides of each equality describe the same set. Working through these will strengthen your ability to translate between set operations and their complements.*

Let A and B be two arbitrary sets in the universe U. Prove each of the following equalities:

(a) $U^c = \emptyset$

(b) $A \cap A^c = \emptyset$

(c) $A \cup A^c = U$

(d) (De Morgan's law) $(A \cup B)^c = A^c \cap B^c$

(e) (De Morgan's law) $(A \cap B)^c = A^c \cup B^c$

(Hints on page 204. Solutions on page 363.)

Exercise 37 (Union-Complement Form)**.** *In this exercise, you'll practice rewriting set expressions in different forms.*

Express the following set expressions using only unions and complements:

(a) $A \setminus (B \cap C)$

(b) $(A \setminus B) \cap (C \setminus D)$

(Hints on page 205. Solutions on page 364.)

Exercise 38 (Symmetric Difference)**.** *In this exercise, you'll work with the symmetric difference of two sets. When you encounter a new definition, it's important to start with what you already know and explore examples and non-examples to build intuition.*
Define $A \triangle B = (A \setminus B) \cup (B \setminus A)$.

(a) Compute $A \triangle B$ for $A = \{1, 2, 3\}$, $B = \{3, 4, 5\}$.

(b) Compute $B \triangle A$ for $A = \{1, 2, 3\}$, $B = \{3, 4, 5\}$.

(c) Show that $\triangle$ is commutative. In particular, prove that for any sets A and B, we have

$$A \triangle B = B \triangle A.$$

(d) Explain in words what elements are contained in $A \triangle B$.

(e) Show that $A \triangle B = (A \cup B) \setminus (A \cap B)$.

(Hints on page 206. Solutions on page 365.)

Chapter 7

The Powerset

Exercise 39 (Power Set Definition)**.** *This exercise will help you practice the distinction between elements and subsets, as well as the definition of the power set.*

Recall the definition of the power set from the textbook:

> "If S is a set, then the power set of S, denoted $\mathcal{P}(S)$, is the set of subsets of S."

Consider the set $A = \{1, 2\}$

(a) Find $\mathcal{P}(A)$.

(b) For each of the following statements, determine whether it is true or false. Explain your reasoning briefly.

  (i) $\mathcal{P}(A) \subseteq A$.

  (ii) $\emptyset \subseteq A$.

  (iii) $\emptyset \subseteq \mathcal{P}(A)$.

  (iv) $\emptyset \in A$.

  (v) $\emptyset \in \mathcal{P}(A)$.

  (vi) $1 \in A$.

  (vii) $1 \in \mathcal{P}(A)$.

  (viii) $\{1\} \in A$.

  (ix) $\{1\} \in \mathcal{P}(A)$.

  (x) $\{1\} \subseteq A$.

  (xi) $\{1\} \subseteq \mathcal{P}(A)$.

(Hints on page 207. Solutions on page 366.)

Exercise 40 (Power Set Computation). *The goal of this exercise is to practice computing the power set of various sets.*

For each of the following sets, find its power set.

(a) $A = \{a\}$

(b) $B = \{a, b\}$

(c) $C = \{a, \{b\}\}$

(d) $D = \{\emptyset, \{\emptyset\}\}$

(e) $E = \mathcal{P}(A)$, where $A = \{a\}$

(f) $F = \{a, b, c\}$

(Hints on page 208. Solutions on page 367.)

Exercise 41 (Power Set Cardinality)**.** *This exercise examines the important relationship between the number of elements in a set and the number of elements in its power set.*

For each set S in the previous exercise, Exercise 40, how many elements are there in S and how many elements are in its power set $\mathcal{P}(S)$? Can you predict how many elements there will be in the power set of $G = \{1, 2, 3, 4\}$? Can you predict how many elements there will be in the power set of $K = \{1, 2, 3, \ldots, k\}$, where $k \in \mathbb{N}$?

For your convenience, the sets from Exercise 40 are listed below.

(a) $A = \{a\}$

(b) $B = \{a, b\}$

(c) $C = \{a, \{b\}\}$

(d) $D = \{\emptyset, \{\emptyset\}\}$

(e) $E = \mathcal{P}(A)$, where $A = \{a\}$

(f) $F = \{a, b, c\}$

(Hints on page 209. Solutions on page 368.)

Exercise 42 (Possible Power Sets)**.** *The goal of this exercise is to help you start thinking more systematically about the special structure of power sets.*

For each set below, determine whether if can can be the power set $\mathcal{P}(S)$ of a set S. If it can be, find the set S. If not, explain why.

(a) $\{1\}$

(b) $\emptyset$

(c) $\{\emptyset, \{1\}\}$

(d) $\{\emptyset\}$

(e) $\{\emptyset, \{1\}, \{\emptyset, 1\}\}$

(f) $\{\emptyset, \{1\}, \{2\}\}$

(g) $\{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$

(Hints on page 210. Solutions on page 369.)

Exercise 43 (Power Sets Closures)**.** *The goal of this exercise is to help you start thinking more systematically about the special structure of power sets.*

Prove that the power set is closed under unions, intersections, and complements, as well as under taking subsets (sometimes called "downward closed"). That is, suppose that S is some set.

(a) Prove that if $X, Y \in \mathcal{P}(S)$ then $X \cup Y \in \mathcal{P}(S)$.

(b) Prove that if $X, Y \in \mathcal{P}(S)$ then $X \cap Y \in \mathcal{P}(S)$.

(c) Prove that if $X \in \mathcal{P}(S)$, then $X^c := S \setminus X \in \mathcal{P}(S)$

(d) Prove that if $X \in \mathcal{P}(S)$ and $Y \subseteq X$, then $Y \in \mathcal{P}(S)$.

(e) Let us revisit the previous exercise, Exercise 42, in light of what we've learned. Can you prove that $\{\{1\}\}$ is not a power set? How about $\{\emptyset, \{1\}, \{2\}\}$?

(Hints on page 211. Solutions on page 370.)

Exercise 44 (Set Operations and Power Sets)**.** *The purpose of this exercise is to study how different set operations on the base sets affect the power set.*

Let U be a fixed "universe" set and $A, B \subseteq U$ sets in this universe. For each statement below, decide whether it is true or false and give a brief justification.

(a) $\emptyset \in \mathcal{P}(A)$.

(b) If $X \in \mathcal{P}(A)$, then $X \in A$.

(c) If $A \subseteq B$, then $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

(d) $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$.

(e) $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$.

(f) $\mathcal{P}(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(B)$.

(g) $\mathcal{P}(A^c) = (\mathcal{P}(A))^c$, where the complement of $\mathcal{P}(A)$ is taken inside $\mathcal{P}(U)$.

(h) $\mathcal{P}(A) \cap \mathcal{P}(A^c) = \{\emptyset\}$.

(Hints on page 212. Solutions on page 371.)

Chapter 8

Index Sets

Exercise 45 (Finite Indices)**.** *This exercise illustrates the relationship between the new "big" symbols and the familiar "small" symbols.*
Consider the sets

$$S_1 := \{1, 2\}; \qquad S_2 := \{2, 3\}; \qquad S_3 := \{3, 4\}.$$

Compute the following sets:

(a) $\bigcup_{i=1}^{3} S_i$.

(b) $\bigcap_{i=1}^{2} S_i$.

(c) $\bigcap_{i=1}^{3} S_i$.

(Hints on page 213. Solutions on page 373.)

Exercise 46 (Infinite Indices). *This exercise demonstrates the utility of the "big" symbols for infinite sequences.*

For each $n \in \mathbb{N}$ let

$$S_n := \left[0, \frac{n-1}{n}\right].$$

(a) Compute the first three sets in the sequence: S_1, S_2, S_3.

(b) Prove that if m, n are natural numbers such that $m < n$ then $S_m \subseteq S_n$. Start by rewriting this claim using mathematical notation.

(c) Find $\bigcap_{i=1}^{\infty} S_i$ and prove your answer.

(d) Find $\bigcup_{i=1}^{\infty} S_i$ and prove your answer. You may use without proof the statement from Problem 2.70: "If $\varepsilon > 0$, then there exists $N \in \mathbb{N}$ such that $1/N < \varepsilon$."

(Hints on page 214. Solutions on page 374.)

Exercise 47 (Uncountable Unions). *This exercise will help you practice using the "big union" symbol to compute with uncountably large collections.*

For each $r \in \mathbb{R}$, define the set S_r as below. Compute $\bigcup_{r\in\mathbb{R}} S_r$. You do not need to prove your answers.

(i) $S_r := \{-r\}$.

(ii) $S_r := \{r^2\}$.

(iii) $S_r := \{e^r\}$, where e is the base of the natural logarithm, $e \approx 2.718$.

(iv) $S_r := \{1\}$.

(v) $S_r := \{r, r^2, -r\}$.

(vi) $S_r := \{1, r\}$.

(vii) $S_r := [0, |r|)$, where $|r|$ is the absolute value of r.

(viii) $S_r := [-|r|, |r|]$.

(ix) $S_r := (-|r|, |r|)$.

(x) $S_r := \{rm : m \in \mathbb{N}\}$.

(Hints on page 215. Solutions on page 376.)

Exercise 48 (Uncountable Unions Revisited)**.** *This exercise will help you practice the definition of the "big union" symbol.*

For each $r \in \mathbb{R}$ define

$$S_r := \{r^2\} \qquad\qquad T_r := \{r^3\}.$$

(a) Use the definition of the "big union" to prove $\bigcup_{r\in\mathbb{R}} S_r \subseteq [0,\infty)$.

(b) Use the definition of the "big union" to prove that $[0,\infty) \subseteq \bigcup_{r\in\mathbb{R}} S_r$. Conclude that

$$\bigcup_{r\in\mathbb{R}} S_r = [0,\infty).$$

(c) Compute $\bigcup_{r\in\mathbb{R}} T_r$ and prove that answer is correct.

(Hints on page 216. Solutions on page 377.)

Exercise 49 (Uncountable Intersections)**.** *This exercise will help you practice using the "big intersection" symbol to compute with uncountably large collections.*

For each $r \in \mathbb{R}$, define the set S_r as below. Compute $\bigcap_{r\in\mathbb{R}} S_r$. You do not need to prove your answers.

(i) $S_r := \{r\}$.

(ii) $S_r := [-|r|, |r|]$, where $|r|$ is the absolute value of r.

(iii) $S_r := (-|r|, |r|)$.

(iv) $S_r := [0, |r|]$.

(v) $S_r := (0, |r|)$.

(vi) $S_r := (-1 - |r|, 1 + |r|)$.

(vii) $S_r := \{m + r \,:\, m \in \mathbb{Z}\}$.

(Hints on page 217. Solutions on page 378.)

Exercise 50 (Uncountable Intersections Revisited)**.** *This exercise will help you practice the definition of the "big intersection" symbol.*

For each $r \in \mathbb{R}$, define

$$S_r := [-|r|, |r|] \qquad\qquad T_r := (-1 - |r|, 1 + |r|)\,.$$

(a) Use the definition of "big intersection" to prove $\bigcap_{r\in\mathbb{R}} S_r \subseteq \{0\}$.

(b) Use the definition of the "big intersection" to prove that $\{0\} \subseteq \bigcap_{r\in\mathbb{R}} S_r$. Conclude that

$$\bigcap_{r\in\mathbb{R}} S_r = \{0\}.$$

(c) Compute $\bigcap_{r\in\mathbb{R}} T_r$ and prove that answer is correct.

(Hints on page 218. Solutions on page 379.)

Exercise 51 (Unions and Intersections I)**.** *The following sequence of exercises is more challenging "capstone exercises". They will help you sharpen your skills working with more complex unions and intersections of infinite sequences and includes mixing these two operations.*

For each $n \in \mathbb{N}$ define the interval

$$I_n := \left[\frac{(-1)^n}{n}, 2 + \frac{1}{n}\right].$$

(a) What are the intervals I_1, I_2, I_3, I_4?

(b) If $k \in \mathbb{N}$ is a natural number, what is I_{2k}? What about I_{2k+1}?

(c) For every natural number $k \in \mathbb{N}$, define $J_k := \bigcap_{n=k}^{\infty} I_{2n}$. Prove that $\left[\frac{1}{2k}, 2\right] \subseteq J_k$.

(d) Prove that $J_k \subseteq \left[\frac{1}{2k}, \frac{4k+1}{2k}\right]$.

(Hints on page 219. Solutions on page 380.)

Exercise 52 (Unions and Intersections II)**.** *This is a continuation of Exercise 51 above.*
Recall that we have defined, for each $n \in \mathbb{N}$, the intervals

$$I_n := \left[\frac{(-1)^n}{n}, 2 + \frac{1}{n}\right].$$

Moreover, for every natural number $k \in \mathbb{N}$, we have defined

$$J_k := \bigcap_{n=k}^{\infty} I_{2n}.$$

(a) Use proof by contradiction to prove that $(x \in J_k) \implies x \leq 2$. Conclude that $J_k \subseteq \left[\frac{1}{2k}, 2\right]$ and therefore

$$J_k = \left[\frac{1}{2k}, 2\right].$$

You may use without proof the statement from Problem 2.70: "If $\varepsilon > 0$, then there exists $N \in \mathbb{N}$ such that $1/N < \varepsilon$."

(b) For every natural number $k \in \mathbb{N}$, define $J'_k := \bigcap_{n=k}^{\infty} I_{2n+1}$. Compute J'_k. You do not need to prove your answer (but you are encouraged to do so).

(c) For every natural number $k \in \mathbb{N}$, define $E_k := \bigcap_{n=k}^{\infty} I_n$. Compute E_k. You do not need to prove your answer.

(d) Compute $\bigcup_{k=1}^{\infty} E_k$. Prove your answer.

(Hints on page 220. Solutions on page 381.)

Exercise 53 (Unions and Intersections III)**.** *This is a continuation of Exercise 51 and Exercise 52 above.*

Recall that we have defined, for each $n \in \mathbb{N}$, the intervals

$$I_n := \left[\frac{(-1)^n}{n}, 2 + \frac{1}{n}\right].$$

Find, with proof, the value of

$$\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} I_n.$$

(Hints on page 221. Solutions on page 382.)

Exercise 54 (Monotone Sequences)**.** *This exercise generalizes Problem 3.36 from the textbook as well as some of the exercises from this handout.*

A sequence of sets $\{S_n\}_{n=1}^{\infty}$ is said to be *increasing* if

$$S_1 \subseteq S_2 \subseteq S_3 \subseteq \cdots .$$

Formally, this means that $\forall m, n \in \mathbb{N}[(m < n) \implies (S_m \subseteq S_n)]$. Similarly, the sequence is said to be *decreasing* if

$$S_1 \supseteq S_2 \supseteq S_3 \supseteq \cdots .$$

Formally, this means that $\forall m, n \in \mathbb{N}[(m < n) \implies (S_m \supseteq S_n)]$.

(a) Suppose $\{S_n\}_{n=1}^{\infty}$ is an increasing sequence. Find, with proof, $\bigcap_{i=1}^{\infty} S_i$.

(b) Suppose $\{S_n\}_{n=1}^{\infty}$ is a decreasing sequence. Formulate a guess as to what $\bigcup_{n=1}^{\infty} S_i$ is.

(c) Prove that $\{S_n\}_{n=1}^{\infty}$ is an increasing sequence if and only if $\{S_n^c\}_{n=1}^{\infty}$ is a decreasing sequence.

(d) Suppose $\{S_n\}_{n=1}^{\infty}$ is a decreasing sequence. Use the generalized DeMorgan Laws to compute $\bigcup_{n=1}^{\infty} S_i$.

(Hints on page 222. Solutions on page 383.)

Exercise 55 (Pairwise Disjoint). *This exercise is more challenging; it will help you practice the definition of pairwise disjoint sets, as well as several proof techniques and notions from Chapter 2.*

For each $n \in \mathbb{N}$, let

$$S_n := \left\{ \frac{1}{n} + m \,:\, m \in \mathbb{Z} \right\}.$$

Use proof by contradiction to show that the collection $\{S_n\}_{n \in \mathbb{N}}$ is pairwise disjoint.

(Hints on page 223. Solutions on page 384.)

Exercise 56 (Limits)**.** *This exercise is more challenging; it is a "capstone exercise" which draws on many of the concepts introduced in previous exercises. In addition, it connects the material of this section to important concepts from Real Analysis (the theory of Calculus). Note that Exercises 51–53 are a special case and it may help to refer back to them.*

Let $\{S_n\}_{n\in\mathbb{N}}$ be a sequence of sets. Define the *limit inferior*

$$\liminf S_n := \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} S_n$$

and the *limit superior*

$$\limsup S_n := \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} S_n.$$

(a) Prove that

$$\liminf S_n = \{x : x \in S_j \text{ for all but finitely many } j \in \mathbb{N}\}.$$

The condition on the right means

$$\exists B \in \mathbb{N}.\forall j \in \mathbb{N}.[(j \geq B) \implies (x \in S_j)].$$

(b) Prove that

$$\limsup S_n = \{x : x \in S_j \text{ for infinitely many } j \in \mathbb{N}\}.$$

The condition on the right means

$$\forall B \in \mathbb{N}.\exists j \in \mathbb{N}.[(j \geq B) \wedge (x \in S_j)].$$

(c) Let us revisit the sets $I_n := \left[\frac{(-1)^n}{n}, 2 + \frac{1}{n}\right]$ from Exercises 51–53. Show that $0 \in \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} I_n$ but $0 \notin \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} I_n$ without computing these sets explicitly.

(d) Prove that $\liminf S_n \subseteq \limsup S_n$.

(e) Suppose $\{S_n\}_{n\in\mathbb{N}}$ is a pairwise disjoint collection. Prove that $\limsup S_n = \liminf S_n$ and find their common value.

(f) Suppose $\{S_n\}_{n\in\mathbb{N}}$ is an increasing sequence (see Exercise 54). Prove that $\limsup S_n = \liminf S_n$ and find their common value.

(g) Suppose $\{S_n\}_{n\in\mathbb{N}}$ is a decreasing sequence (see Exercise 54). Prove that $\limsup S_n = \liminf S_n$ and find their common value.

(h) Suppose $\{S_n\}_{n\in\mathbb{N}}$ is a sequence such that $S_1 = S_3 = S_5 = \cdots$ and also $S_2 = S_4 = S_6 = \cdots$. Compute $\limsup S_n$ and $\liminf S_n$.

(Hints on page 224. Solutions on page 385.)

Chapter 9

Cartesian Product of Sets

Exercise 57 (Tuples vs. Sets)**.** *The goal of this exercise is to reflect on the difference between sets and tuples via the definition of equality between these mathematical constructs.*

Suppose a, b are two different natural numbers. Explain in your own words why $\{a, b\} = \{b, a\}$ but $(a, b) \neq (b, a)$. Next, use the definition of equality to prove that $\{a, b\} = \{b, a\}$ but $(a, b) \neq (a, b)$.

(Hints on page 225. Solutions on page 389.)

Exercise 58 (Computing Products)**.** *This exercise will help you practice the definition of the Cartesian product of sets.*

(a) Let $A = \{0, 1\}$ and $B = \{-1, 1\}$. Determine which of the following tuples is an element of $A \times B$:

  (i) $(0, -1)$;
  (ii) $(1, 2)$;
  (iii) $(-1, 1)$;
  (iv) $(1, 1)$;
  (v) $(0, 0)$.

(b) Let $A = \{1, 2\}$ and $B = \{3, 4, 5\}$.

  (i) List all elements in the set $A \times B$.
  (ii) List all elements in the set $B \times A$.
  (iii) Is $A \times B = B \times A$?

(c) Let $A = \{x, y, z\}$ and $B = \{1\}$. List all element of $A \times B$.

(d) Let A be an arbitrary set and $B = \{b\}$. Use set-builder notation to describe the set $A \times B$.

(Hints on page 226. Solutions on page 390.)

Exercise 59 (Empty Products)**.** *This exercise will help you practice the definition of the Cartesian product of sets.*

Prove that $A \times B = \emptyset$ if and only if one of A, B is the empty set. That is,

(a) Prove that if $A = \emptyset$ then $A \times B = \emptyset$.

(b) Prove that if $B = \emptyset$ then $A \times B = \emptyset$.

(c) Prove that if $A \times B = \emptyset$, then $A = \emptyset$ or $B = \emptyset$.

(Hints on page 227. Solutions on page 391.)

Exercise 60 (Properties of Cartesian Products)**.** *In this exercise we study the algebraic properties of the product operation on sets.*

While the Cartesian product is called a "product", it does not necessarily resemble other product operations such as multiplication of integers (or of matrices).

(a) **Commutativity.** Give an example of sets A, B for which $A \times B \neq B \times A$.

(b) Give an example of sets A, B for which $A \times B = B \times A$. Can you formulate a hypothesis for which sets $A \times B = B \times A$? (We shall prove such a criterion in the next exercise.)

(c) **Associativity.** Prove that if $A, B, C \neq \emptyset$ then $(A \times B) \times C \neq A \times (B \times C)$. What happens if one of the sets is empty?

(d) **Cancellation.** Prove that if $A \neq \emptyset$ and $A \times B = A \times C$, then $B = C$. Give an example to show the conclusion can fail if $A = \emptyset$.

(Hints on page 228. Solutions on page 392.)

Exercise 61 (Criteria for Commutativity). *In this exercise we study conditions under which the Cartesian product is commutative.* Let A, B, C, D be sets.

(a) Suppose that $A \subseteq C$ and $B \subseteq D$. Prove that $A \times B \subseteq C \times D$.

(b) Suppose $A, B \neq \emptyset$. Prove that $(A \times B \subseteq C \times D) \implies [(A \subseteq C) \wedge (B \subseteq D)]$. Give an example to show that the conclusion may fail if one of A, B is empty.

(c) Prove that $A \times B = B \times A$ if and only if: $A = B$ or one of A, B is empty.

(Hints on page 229. Solutions on page 393.)

Exercise 62 (Product Projections)**.** *This exercise is slightly more challenging; it will help you practice many of the definitions and concepts we've seen so far (set comprehension, quantifiers, Cartesian products), as well as set the stage for some of the material in Chapter 7.*

Let A, B be sets and $S \subseteq A \times B$ some subset of pairs. Define the *projection of S on A*, denoted $\pi_A(S)$ by

$$\pi_A(S) := \{a \in A : \exists b \in B\,[(a, b) \in S]\}.$$

Similarly, define the *projection of S on B*, denoted $\pi_B(S)$ by

$$\pi_B(S) := \{b \in B : \exists a \in A\,[(a, b) \in S]\}.$$

(a) Consider $A = \{1, 2\}$ and $B = \{3, 4\}$. What is the size of $A \times B$? How many different choices are there for $S \subseteq A \times B$? What are the possible sizes of S?

(b) Choose an S for each possible size and compute $\pi_A(S)$ as well as $\pi_B(S)$.

(c) Is it always true that (for any choice of A, B and $S \subseteq A \times B$) $\pi_A(S) \times \pi_B(S) = S$?

(d) Can you find conditions on A, B and $S \subseteq A \times B$ under which $\pi_A(S) \times \pi_B(S) = S$? Try to make your condition as general as possible. For an extra challenge and practice, try to prove your answer.

(Hints on page 230. Solutions on page 394.)

Exercise 63 (Visualizing Products). *In this exercise we use the Cartesian plane to help us visualize Cartesian products of sets.*

Sketch each Cartesian product below as part of the Cartesian plane $\mathbb{R}^2$ and describe the relevant region in words. Pay special attention to the boundary of the region.

(a) $(0, 1) \times [2, 3]$.

(b) $(-\infty, 0] \times [0, \infty)$.

(c) $\mathbb{R} \times \mathbb{N}$; is this the same set as $\mathbb{N} \times \mathbb{R}$?

(Hints on page 231. Solutions on page 395.)

Exercise 64 (Distributivity of the Product I)**.** *In this exercise we start investigating the distributivity of the Cartesian product; that is, how it relates to other set operations.*
Let $A = \{0, 1\}$, $B = \{2, 3\}$, and $C = \{3, 4\}$.

(a) Compute the sets $A \times (B \cup C)$ and $(A \times B) \cup (A \times C)$.

(b) Compute the sets $A \times (B \cap C)$ and $(A \times B) \cap (A \times C)$.

(c) Compute the sets $A \times (B \setminus C)$ and $(A \times B) \setminus (A \times C)$.

(Hints on page 232. Solutions on page 396.)

Exercise 65 (Distributivity of the Product II)**.** *In this exercise we prove rules generalizing our results from the previous exercise about how the Cartesian product relates to other set operations.*

In Exercise 60, we saw that the Cartesian product lacks some properties of regular products (such as commutativity and associativity) but also enjoys some of the same properties of regular products (such as canceling). In this exercise we show that the Cartesian product enjoys extensive distributivity properties.

Let A, B, C be arbitrary sets (which may or may not be empty).

(a) Distributivity of product over union. Prove that $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

(b) Distributivity of product over intersection. Prove that $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

(c) Distributivity of product over set-difference. Prove that $A \times (B \setminus C) = (A \times B) \setminus (A \times C)$.

(d) Distributivity of product over symmetric difference. Prove that $A \times (B \triangle C) = (A \times B) \triangle (A \times C)$. Recall that $\triangle$ is the *symmetric difference* of sets:

$$X \triangle Y = (X \setminus Y) \cup (Y \setminus X) = (X \cup Y) \setminus (X \cap Y).$$

(e) Does the product distribute "from the right" as well as "from the left"? By this we mean, is it true that $(A \cup B) \times C = (A \times C) \cup (B \times C)$? What about if "$\cup$" is replaced by "$\cap$" or "$\setminus$" or "$\triangle$"?

(Hints on page 233. Solutions on page 397.)

Exercise 66 (Product and Other Set Operations)**.** *This exercise investigates how two (or more) Cartesian products behave under the set operations of union and intersection.*

Let A, B, C, D be sets.

(a) Prove that $(A \times C) \cap (B \times D) = (A \cap B) \times (C \cap D)$.

(b) Give a counterexample to show that $(A \times C) \cup (B \times D) \neq (A \cup B) \times (C \cup D)$.

(c) Suppose A, B are nonempty and disjoint and that $(A \times C) \cup (B \times D) = (A \cup B) \times (C \cup D)$. What can you conclude about the sets C, D?

(Hints on page 234. Solutions on page 398.)

Exercise 67 (Product and Other Set Operations II)**.** *This exercise investigates how two (or more) Cartesian products behave under set complementation.*

Suppose that U, V are some "universes of sets" and that $X \subseteq U$ and $Y \subseteq V$, so that $X^c = U \setminus X$ and $Y^c = V \setminus Y$. Under these conditions we have $X \times Y \subseteq U \times V$ so that $(X \times Y)^c = (U \times V) \setminus (X \times Y)$.

(a) Give an example to show that $(X \times Y)^c \neq X^c \times Y^c$.

(b) Prove that $(X \times Y)^c = (X^c \times V) \cup (U \times Y^c)$.

(Hints on page 235. Solutions on page 399.)

Exercise 68 (Distributivity Revisited)**.** *This exercise generalizes Exercise 65.*

Recall that arbitrary unions and intersections ("big symbols") were introduced in the previous section. Let us generalize our results from Exercise 65. Let I be an arbitrary nonempty index set (possibly infinite), $\{A_i\}_{i\in I}$ a collection of sets indexed by I, and B an arbitrary set.

(a) Prove that $\left(\bigcup_{i\in I} A_i\right) \times B = \bigcup_{i\in I} (A_i \times B)$.

(b) Prove that $\left(\bigcap_{i\in I} A_i\right) \times B = \bigcap_{i\in I} (A_i \times B)$.

(c) What if the "big symbols" were to appear "on the right", as in $B \times \left(\bigcup_{i\in I} A_i\right)$?

(Hints on page 236. Solutions on page 400.)

Chapter 10

Introduction to Mathematical Induction

Exercise 69 (Inductive reasoning)**.** *The goal of this exercise is to reflect on the axiom of induction and why it may be plausible.*

A metaphor that is often used for induction is a chain of dominoes: if the first one falls, and if each domino knocks over the next one, then they all fall!

(a) Let $P : \mathbb{N} \to \{\text{True}, \text{False}\}$ be a predicate on the natural numbers. Suppose $P(1)$ is true, and also $(\forall n \in \mathbb{N})(P(n) \implies P(n+1))$. Prove $P(2), P(3), P(4)$.

(b) How would you go about proving that $P(100)$ is true? You're not asked to actually prove it!

(c) Can you explain informally why, from the two assumptions above, it is plausible to conclude that $P(n)$ is true for **all** natural numbers n?

(d) Here is a tough question worth reflecting on carefully: why do we need a whole new axiom of induction? Can we not prove directly that $(\forall n \in \mathbb{N})P(n)$?

(Hints on page 237. Solutions on page 401.)

Exercise 70 (Recap). *In this exercise, we will study in detail the key example from the textbook. The goal is for you to reconstruct on your own the ideas and logic you've seen in class, which will help you understand them much more deeply. Therefore, for this exercise please do not refer back to the text or your notes!*

The n-th *triangular number*, denoted T_n, is defined by the equation

$$T_n = \frac{n(n+1)}{2}.$$

(a) Compute the first five triangular numbers.

(b) Consider the claim

> The sum of the first n natural numbers is the n-th triangular number.

Use mathematical notation to define a **predicate** $P(n)$ of the form $a = b$ to express this claim. Our goal is to prove $(\forall n \in \mathbb{N})P(n)$ by mathematical induction.

(c) What is $P(1)$? Is it true? This is the **base case** of the induction.

(d) What is $P(n+1)$? The **inductive step** is the proof that $P(n) \implies P(n+1)$. We call $P(n)$ the **inductive hypothesis**.

(e) Use direct proof to show $P(n) \implies P(n+1)$.

(f) Can you summarize the proof pattern we have just used? What are the key steps of an inductive proof?

(Hints on page 238. Solutions on page 402.)

Exercise 71 (Writing inductive proofs)**.** *The goal of this exercise is to practice translating informal mathematical statements into formal statements and use the inductive framework to prove them.*

In this exercise we will prove that

> The sum of the first n odd natural numbers is the n-th square number.

(a) Verify the first five cases of the claim.

(b) Define a *predicate* $P(n)$ of the form $a = b$ to express the claim mathematically.

(c) What is $P(1)$?

(d) What is $P(n+1)$?

(e) Use direct proof to prove $P(n) \implies P(n+1)$.

(Hints on page 239. Solutions on page 403.)

Exercise 72 (Writing inductive proofs II). *In this exercise you get to practice everything we've learned in this chapter. We present a problem as it would appear on an exam and ask you to write a complete proof!*

Use mathematical induction to prove that for every $n \in \mathbb{N}$,

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}.$$

Make sure to define a predicate and clearly explain the steps in your proof!

(Hints on page 240. Solutions on page 404.)

Exercise 73 (False proof)**.** *In this exercise you will play the role of peer-reviewer or a grader and criticize an inductive argument.*

Define the predicate $P(n)$ by

$$1 + 2 + 4 + \cdots + 2^n = 2^{n+1} + 1.$$

(a) What is the assertion $P(n+1)$?

(b) Use direct proof to show that $\forall n \in \mathbb{N}(P(n) \implies P(n+1))$.

(c) What is $P(3)$? Is it true?

(d) Why haven't we shown $\forall n \in \mathbb{N}\, P(n)$? Is induction wrong after all?

(e) Challenge: Can you "correct" the claim? That is, can you find an predicate very similar to the above which is true for every natural number?

(Hints on page 241. Solutions on page 405.)

Chapter 11

More Mathematical Induction

Exercise 74 (Asymptotic growth)**.** *This exercise shows how induction can be used to prove statements that are "eventually" true.*

Recall that $n!$ (read "n factorial") is defined as the product of the first n integers $n! = 1 \cdot 2 \cdot 3 \cdots n$. Use mathematical induction to prove that *eventually* the following inequality holds.

$$n! > 2^n.$$

Remember to define a predicate and clearly label the inductive hypothesis and where it is used.

(Hints on page 242. Solutions on page 406.)

Exercise 75 (Asymptotic growth II)**.** *This exercise seems similar to the previous one, but the inductive step involves one more idea than simply substituting, so be alert!*

Use mathematical induction to prove that *eventually* the following inequality holds.

$$n^n > n!$$

Remember to define a predicate and clearly label the inductive hypothesis and where it is used.

(Hints on page 243. Solutions on page 407.)

Exercise 76 (Convergence)**.** *Induction is a widespread technique which is used throughout mathematics, not only to prove properties of natural numbers. This exercise shows how induction can be used in analysis (calculus) to help with limits.*

Recall that $n! = n \cdot (n-1) \cdot (n-2) \cdots 1$. A variation on the factorial notation is the double-factorial, which means that each multiplicand skips two numbers:

$$n!! = n \cdot (n-2) \cdot (n-4) \cdots a$$

where a is the smallest natural number for which this sequence makes sense. That is, $a = 1$ if n is odd and $a = 2$ if n is even.[1]

(a) Compute $n!!$ for $n = 1, 2, \ldots, 10$.

(b) With the factorial we have the recursive definition $(n+1)! = (n+1) \cdot n!$ (with the starting condition $n = 1$). Can you find a similar recursive definition for the double factorial?

(c) We define the sequence $a_n = \frac{(2n-1)!!}{(2n)!!}$. Compute the first five terms of the sequence.

(d) Derive a recurrence for a_{n+1} in terms of a_n.

(e) Use mathematical induction to prove

$$\frac{1}{\sqrt{4n}} \leq a_n \leq \frac{1}{\sqrt{2n+1}}.$$

Conclude that a_n converges to 0.

(Hints on page 244. Solutions on page 408.)

[1] Be careful to distinguish $n!!$ and $(n!)!$. In particular, note that $n!! < n! < (n!)!$ for every $n \geq 2$.

Exercise 77 (Convergence II)**.** *Here is another example where induction can be used to prove convergence, this time of an infinite product! The exercise also demonstrates a common phenomenon of different patterns for odd and even terms.*

Consider the following sequence $\{p_n\}_{n=1}^{\infty}$ whose n-th term is defined by the product

$$p_n = \prod_{k=1}^{n} \left(1 + \frac{(-1)^n}{n+1}\right).$$

(a) Compute the first six terms of the sequence.

(b) Do you see a pattern in your computation? Conjecture a formula for the odd terms p_{2n-1} and for the even terms p_{2n}.

(c) Use mathematical induction to prove your conjecture about the odd terms p_{2n+1}.

(d) Prove your conjecture about the even terms p_{2n}.

(e) Conclude that the infinite product below converges and find its limit

$$\prod_{k=1}^{\infty} \left(1 + \frac{(-1)^n}{n+1}\right).$$

(Hints on page 245. Solutions on page 410.)

Chapter 12

Complete Induction

Exercise 78 (Recurrence)**.** *A famous variation on the Fibonacci sequence is called the Lucas sequence. Its recurrence formula is the same, it's only the starting conditions that are different. Therefore, the Lucas sequence enjoys many of the same properties that the Fibonacci sequence has. For example, compare this exercise (and its proof!) to Problem 4.29 from the recommended reading.*

Let $a_1 = 1$, $a_2 = 3$, and $a_n = a_{n-1} + a_{n-2}$ for $n \geq 3$. Prove by induction that $a_n < 2^n$ for all n.

Remember to clearly define your predicate and prove the base case(s)!

(Hints on page 246. Solutions on page 412.)

Exercise 79 (Remainder modulo 3). *This exercise is a particular case of the Division Algorithm, also known as the Quotient-Remainder formula.*

Use complete induction to prove that every integer $n \geq 2$ can be expressed as $n = 3q + r$ with $q, r \in \mathbb{Z}_{\geq 0}$ and $r \in \{0, 1, 2\}$.

Remember to clearly define your predicate and prove the base case(s)!

(Hints on page 247. Solutions on page 413.)

Exercise 80 (Making change)**.** *This exercise will help you practice strong induction. It is a variation on Problem 4.31 from the recommended reading.*

Suppose you have an infinite amount of \$6, \$10, and \$15 bills. Prove that any whole number of dollars greater or equal to \$30 can be made exactly (no change required). For example, to pay \$30 one could take five \$6 bills or three \$10 bills, or two \$15 bills.

Remember to clearly define your predicate and prove the base case(s)!

(Hints on page 248. Solutions on page 414.)

Exercise 81 (Fibonacci)**.** *One of the many uses of induction is for proving a closed-form formula for recursive relations. Here is a famous such formula for the Fibonacci sequence (cf. Problem 4.29 in the recommended reading).*

Recall that the Fibonacci sequence is given by the linear recurrence $f_1 = f_2 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$. The first few values of the Fibonacci sequence are

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \ldots$$

Use complete induction to prove that

$$f_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}.$$

Remember to clearly define your predicate and prove the base case(s)!

(Hints on page 249. Solutions on page 415.)

Exercise 82 (Divisibility)**.** *The goal of this exercise is to prove an inductive statement (and a useful fact!) which holds only for some natural numbers, which are not necessarily consecutive. It may not be immediately apparent how to use the inductive hypothesis.*

Let $a, b \in \mathbb{N}$ be two constant natural numbers (we don't know which). Prove by mathematical induction that for every *odd natural* n the number $a + b$ divides $a^n + b^n$.

(Hints on page 250. Solutions on page 417.)

Chapter 13

The Well-Ordering Principle

Exercise 83 (Maximum and Minimum)**.** *This exercise will help you practice the definitions of minimum and maximum, which are important throughout mathematics and essential for the well-ordering principle.*
Find the maximum and minimum, if they exist, of each of the following sets. Use the definition of maximum and minimum to prove your answers.

(a) $\mathbb{N}$

(b) $\mathbb{Z}$

(c) $\emptyset$

(d) $A = \{n \in \mathbb{N} : n \text{ is a multiple of } 3\}$

(e) $B = \{z \in \mathbb{Z} : z > 11\}$

(f) $C = \{r \in \mathbb{R} : 0 < r < 1\} = (0, 1)$

(g) $D = \{x \in \mathbb{R} : 0 \leq x \leq 1\} = [0, 1]$

(Hints on page 251. Solutions on page 418.)

Exercise 84 (Spot the error)**.** *This exercise will help you practice the statement of the well-ordering principle. It points out some common errors that are often seen on solutions, can you identify and correct them?*

Identify the error in each statement below and give a set that serves as a counterexample.

(a) Every subset of $\mathbb{N}$ has a least element.

(b) Every nonempty subset of $\mathbb{Z}$ has a least element.

(c) Every nonempty subset of $\mathbb{N}$ has a greatest element.

(Hints on page 252. Solutions on page 420.)

Exercise 85 (Well-Ordering from Induction)**.** *The recommended reading gave a proof outline for deriving the Well-Ordering Principle from Induction. In this exercise we develop this proof carefully.*

We will show that the principle of mathematical induction implies the Well-Ordering principle.

(a) For the sake of contradiction, suppose that S is a nonempty subset of $\mathbb{N}$ that does not have a least element. Define the predicate $P(n) := n \notin S$. Use induction to prove $\forall n \in \mathbb{N}.P(n)$.

(b) Why do we have a contradiction? Conclude that S must have a least element.

You have now proved that if the principle of mathematical induction holds, then the well-ordering principle holds. (Can you explain why?)

(Hints on page 253. Solutions on page 421.)

Exercise 86 (Induction from Well-Ordering)**.** *The recommended reading asserts that induction and well-ordering are equivalent principles. In Exercise 85 we have shown that induction implies well-ordering. In this exercise you will prove that well-ordering implies induction.*

We prove that the Well-Ordering Principle implies the Axiom of Induction.

Let $S \subseteq \mathbb{N}$ such that

- $1 \in S$,
- $\forall n \in \mathbb{N}.[(n \in S) \implies (n+1 \in S)]$.

Suppose towards contradiction $S \neq \mathbb{N}$ and consider $S^c = \{n \in \mathbb{N} : n \notin S\}$. Use the well-ordering principle to arrive at a contradiction. Be sure to carefully justify your steps.

You have now proved that if the well-ordering principle holds, then so does the principle of mathematical induction. (Can you explain why?)

(Hints on page 254. Solutions on page 422.)

Exercise 87 (Using the Well-Ordering Principle)**.** *Since the Well-Ordering Principle is equivalent to induction, we should be able to use it in proofs where we normally use inductions. Here is one such example.*

Use the well-ordering principle to prove that

$$2 + 4 + \cdots + 2n = n(n + 1).$$

Do not use induction. (Hints on page 255. Solutions on page 423.)

Exercise 88 (Division with remainder)**.** *Even though they are equivalent, for some proofs it is more convenient to use the well-ordering principle than induction. In the previous handout on complete induction, we have proved a special case of division with remainder. We shall now prove the full theorem.*
Use the well-ordering principle to prove

Theorem (Division with remainder)**.** For every $n, m \in \mathbb{N}$ there exist $q \in \mathbb{Z}_{\geq 0}$ and $r \in \{0, 1, \ldots, m-1\}$ such that $n = qm + r$. Furthermore, these q and r are unique. We call q the **quotient** and r the **remainder** of dividing n by m.

(a) Let $m, n \in \mathbb{N}$ be arbitrary. Consider

$$S = \{x \in \mathbb{Z}_{\geq 0} : \exists q \in \mathbb{Z}_{\geq 0}. x = n - qm\}$$

Prove that S is nonempty.

(b) Let r be the minimal element[1] of S. Prove that $r \in \{0, 1, \ldots, m-1\}$.

(c) Conclude that there exist $q \in \mathbb{Z}_{\geq 0}$ and $r \in \{0, 1, \ldots, m-1\}$ such that $n = qm + r$.

(d) Suppose $q, q' \in \mathbb{Z}_{\geq 0}$ and $r, r' \in \{0, 1, \ldots, m-1\}$ are such that $n = qm + r = q'm + r'$. Prove that $q = q'$ and $r = r'$.

(Hints on page 256. Solutions on page 424.)

[1]Here we are using the generalized well-ordering principle. Note that S is a subset of the integers which is bounded below by 0.

Exercise 89 (Spot the error II)**.** *Just as with induction, when using the well-ordering principle one has to be careful to avoid some common but not-so-obvious false steps. Can you spot the error in the following proof? Recall that the Fibonacci sequence was introduced in Problem 4.29 in the text.*

Recall that the Fibonacci sequence is given by the linear recurrence $f_1 = f_2 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$. The first few values of the Fibonacci sequence are

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \ldots$$

Here is a "proof" that all Fibonacci numbers are even. Can you identify and explain the error in this proof?

Proof attempt. We prove that all Fibonacci numbers are even. Let $S = \{n \in \mathbb{N} : f_n \text{ is odd}\}$ be the set of counter-examples. We prove that S is empty using the well-ordering principle.

Assume for contradiction that S is nonempty, so by the well-ordering principle it has a minimal element, say $s \in S$. Now,

$$f_s = f_{s-1} + f_{s-2}.$$

Since s is the minimal element of S, we know that $s-1, s-2 \notin S$, so that f_{s-1}, f_{s-2} must be even. Therefore, $f_s = f_{s-1} + f_{s-2}$ is also even, contradicting the assumption that $s \in S$.

We supposed S is nonempty and arrived at a contradiction. This contradiction proves that S is empty, so all Fibonacci numbers are even. □

(Hints on page 257. Solutions on page 425.)

Exercise 90 (Roundabout). *Here is a classic result in which a proof via the well-ordering principle is much easier than an inductive proof. Challenge yourself to solve this famous puzzle!*

The mythical country Maths has a finite number of cities which are connected to each other via a finite number of one-way roads in such a way that each city is reachable from any other city (though not necessarily via a direct route).

Prove that it is possible to plan a round-trip (a tour starting and ending at the same city) which does not visit any city more than once (except the starting and ending city is “visited” exactly twice: at the beginning and the end). Note that you can choose the starting and ending point and your tour does not have to visit every city.

(Hints on page 258. Solutions on page 426.)

Chapter 14

Relations

Exercise 91 (Describing Relations). *This exercise illustrates three common ways of depicting relations.* Let $X = \{1, 2, \ldots, 10\}$ and $Y = \{a, b, c, d, e\}$. We define a relation R from X to Y by the following conditions:

- a is related to every element;
- b is related to the even elements and is not related to any odd element;
- $3Rc$, $6Rc$, and $9Rc$;
- $4Rd$ and $8Rd$;
- $5Re$ and $10Re$.

(a) Is $R \subseteq X \times Y$ or $R \subseteq Y \times X$?

(b) Write out the set R.

(c) Draw a directed graph depicting the relation R.

(d) Another common way of depicting a relation is via a *logical table*. The elements of X label the rows and the elements of Y label the columns. A cell in row x and column y has 1 if x is related to y, and 0 otherwise. Complete the logical table for the relation R.

	a	b	c	d	e
1					
2					
3					
4					
5					
6					
7					
8					
9					
10					

(Hints on page 259. Solutions on page 427.)

Exercise 92 (Properties of Relations)**.** *This exercise will help you practice common properties of relations, which we will encounter in future sections as well.*

Let S be the set of all students at the University of Toronto. We define the following relations on S:

aCb iff a has taken more courses than b.
aDb iff a and b are in the same degree program.
aNb iff a and b have no course in common this semester.
aSb iff a and b have at least one course in common.

For each relation determine, with an explanation, which of the properties below does it have.

	reflexive	symmetric	transitive
C			
D			
N			
S			

(Hints on page 260. Solutions on page 428.)

Exercise 93 (Describing Properties of Relations)**.** *This exercise will help you practice common properties of relations and the different ways of representing them.*

Let us consider different properties of relations on the set $A := \{a, b, c, d, e\}$. Two common ways of representing relations on a set are via a logical table (cf. Exercise 91) and via a directed graph.

	a	b	c	d	e
a					
b					
c					
d					
e					

(a) Suppose that R is a reflexive relation on A. How would you fill the logical table to reflect this fact? How would you modify the digraph to reflect this fact?

(b) Suppose you are asked to determine whether a relation is reflexive.

- If the relation is described using a set, how would you determine if it is reflexive?
- If the relation is described using a digraph, how would you determine if it is reflexive?
- If the relation is described using a logical table, how would you determine if it is reflexive?

Would you prefer to receive a description of the relation as a set, a digraph, or a logical table? Explain your reasoning.

(c) Suppose that S is a symmetric relation on A, and that bSb, cSd, and eSa. How would you fill the logical table to reflect these fact? How would you modify the digraph to reflect these facts?

(d) How could you determine if a relation is symmetric using each of the set description, digraph description, and logical table description? Which description would you prefer? Explain your reasoning.

(e) Which of the three common ways of describing a relation would you prefer if you had to determine whether a relation is transitive?

(Hints on page 261. Solutions on page 429.)

Exercise 94 (Counting Relations)**.** *This exercise will help you think through the definition of relations through counting.*

(a) If R is a relation on the set $\{1, 2, 3\}$, then R is a subset of which set?

(b) How many different relations on the set $\{1, 2, 3\}$ are there? What about on the set $\{1, 2, \ldots, n\}$?

(c) If R is a *reflexive* relation on the set $\{1, 2, 3\}$, what elements must R contain?

(d) How many different reflexive relations on the set $\{1, 2, 3\}$ are there? What about on the set $\{1, 2, \ldots, n\}$?

(e) How many different relations on the set $\{1, 2, 3\}$ are both reflexive and symmetric? What about on the set $\{1, 2, \ldots, n\}$?

(f) List all relations on $\{1, 2, 3\}$ that satisfy all of the following three conditions: reflexive, symmetric, and transitive.

(Hints on page 262. Solutions on page 431.)

Exercise 95 (Weak ordering)**.** *This exercise defines order relations, which are ubiquitous throughout mathematics; in fact, you are already familiar with several examples!*

Definition. The relation R on the set A is **antisymmetric** if for all $a, b \in A$, aRb and bRa together imply $a = b$.

Definition. The relation R on the set A is said to be a **(weak) ordering** if it is reflexive, antisymmetric, and transitive.

Prove that each of the following relations is an ordering.

(a) The relation $\leq$ on $\mathbb{N}$.

(b) The relation $\subseteq$ on $\mathcal{P}(\{1, 2, 3\})$.

(c) The divisibility relation on $\mathbb{N}$ (i.e. aRb if and only if $a|b$).

(Hints on page 263. Solutions on page 433.)

Exercise 96 (Strict ordering). *Sometimes mathematician prefer to define ordering in a different way—think of the difference between $\leq$ and $<$. This exercise shows that each definition can be derived from the other.*

Definition. The relation R on the set A is **asymmetric** if for all $a, b \in A$, if aRb, then $\neg(bRa)$.

Definition. The relation R on the set A is said to be a **strict ordering** if it is asymmetric and transitive.

(a) Prove that the $<$ relation on $\mathbb{N}$ is a strict ordering.

(b) Prove that a strict ordering is necessarily irreflexive, i.e. for all $a \in A$, $\neg(aRa)$.

(c) Let R be a weak ordering on A (cf. Exercise 95). Define S on A by aSb if and only if aRb and $a \neq b$. Prove that S is a strict ordering on A.

(d) Let S be a strict ordering on A. Define R on A by aRb if and only if aSb or $a = b$. Prove that R is a weak ordering on A (cf. Exercise 95).

(Hints on page 264. Solutions on page 434.)

Chapter 15

Equivalence Relations

Exercise 97 (Real-world relations)**.** *This exercise will help you practice the definition of equivalence relations and equivalence classes.*

Recall the relations from Exercise 2 of the previous handout. For each of these relations, determine if it is an equivalence relations and if so describe the equivalence classes.

Let S be the set of all students at the University of Toronto. We define the following relations on S:

aCb iff a has taken more courses than b.

aDb iff a and b are in the same degree program.

aNb iff a and b have no course in common this semester.

aSb iff a and b have at least one course in common.

(Hints on page 265. Solutions on page 435.)

Exercise 98 (String length). *Here is further practice on the definition of equivalence relations and equivalence classes, with a "more mathematical" example.*

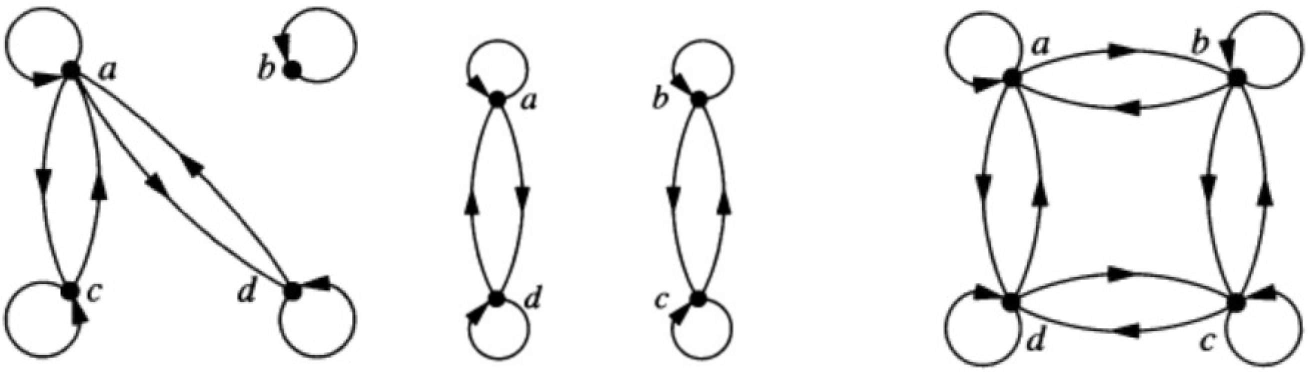
Let S be the set of all words in the English language. For two words $w_1, w_2 \in S$, define $w_1 \sim w_2$ if they have the same length (number of letters). So we would have `cat` $\sim$ `dog`, because each of these words is composed of exactly 3 letters, and so both have the same length. However, `cat` $\not\sim$ `bird`, because `cat` has length 3, while `bird` has length 4.

Prove that $\sim$ is an equivalence relation and describe its equivalence classes.

(Hints on page 266. Solutions on page 436.)

Exercise 99 (Digraphs). *This exercise will help you practice interpreting relations presented as digraphs, as well as the definition of equivalence relations and equivalence classes.*

Determine if the relation with the digraph shown is an equivalence relation. If not, explain what properties fail. If yes, list the equivalence classes.



(a) (b) (c)

(Hints on page 267. Solutions on page 437.)

Exercise 100 (Common misconception)**.** *This exercise asks you to read a short proof and spot the subtle error. This will sharpen your understanding of the definitions, improve your proof-reading and writing skills, as well as your ability to evaluate arguments!*

Read the following "proof" that every symmetric and transitive relation is reflexive (and thus an equivalence relations). Explain the error in the proof and prove that it is an error by constructing a counterexample.

Proof. Let $\sim$ be a relation on a set X that is symmetric and transitive. We prove that $\sim$ is reflexive and therefore an equivalence relation. Let $a \in X$ be arbitrary. For any $b \in X$ such that $a \sim b$ we have by symmetry $b \sim a$. Since $a \sim b$ and $b \sim a$ we have by transitivity $a \sim a$. This proves that $\sim$ is reflexive. □

(Hints on page 268. Solutions on page 438.)

Exercise 101 (Multifunctional)**.** *In this exercise you will prove a new theorem about a new concept! In addition to being excellent practice, you'd be able to use the theorem right away to examine a novel relation.*

A relation R on a set X is called **multifunctional**[1] if

$$\forall x \in X \, \exists y \in X \, (xRy).$$

For example R on the set of real numbers $\mathbb{R}$ defined by xRy if and only if $x^2 = y$ is multifunctional. (Do you see why?).

(a) Prove that if R is multifunctional, symmetric, and transitive, then R is an equivalence relation.

(b) Consider the relation $\sim$ on $M_n(\mathbb{R})$ (the set of all $n \times n$ matrices with real entries) defined by $A \sim B$ if and only if $A - B$ is invertible. Which of the four properties "reflexivity", "symmetry", "transitivity", "multifunctionality" does $\sim$ have? Justify your answer.

(Hints on page 269. Solutions on page 439.)

[1] We will learn a lot more about *functional* relations in Chapter 8. Many of the operations you're familiar with have inverses that are multifunctions rather than functions; an example is the operation of taking a square root, which is the example relation R in the exercise statement. Multifunctions play an important role in Complex Analysis.

Exercise 102 (Remainders)**.** *One of the most fundamental examples of equivalence relations arises from modular arithmetic. We will study this example in much greater detail in §7.4.*

Recall that in the handout on the Well-Ordering Principle you'd proved the division with remainder theorem:

Theorem (Division with remainder)**.** For every $n, m \in \mathbb{N}$ there exist $q \in \mathbb{Z}_{\geq 0}$ and $r \in \{0, 1, \ldots, m-1\}$ such that $n = qm + r$. Furthermore, these q and r are unique. We call q the **quotient** and r the **remainder** of dividing n by m.

Fix some $m \in \mathbb{N}$ and define a relation M on $\mathbb{N}$ by aMb if and only if a and b have the same remainder when divided by m.

(a) Suppose $a \geq b$. Prove that aMb if and only if $m|(b-a)$. Here we are assuming that every natural number divides 0.[2]

(b) Prove that M is an equivalence relation.

(c) What are the equivalence classes for $m = 3$? For $m = 5$? What about $m = 1$?

(d) Define a relation D on $\mathbb{N}$ by aDb if and only if a and b end in the same digit (when written according to the usual way of writing numbers). Prove that D is an equivalence relation.

(Hints on page 270. Solutions on page 440.)

[2]More formally, $x|y$ if and only if $\exists q \in \mathbb{Z}_{\geq 0}.(y = xq)$.

Exercise 103 (Advanced Mathematics). *Equivalence relations are ubiquitous throughout mathematics. This exercise highlights two famous ones that are often studied in advanced courses.*

(a) Consider the relation R on $\mathbb{R}$ defined by xRy if and only if $y - x \in \mathbb{Z}$. Prove that R is an equivalence relation and describe its equivalence classes.

(b) Consider the relation Q on $\mathbb{R}$ defined by xQy if and only if $y - x \in \mathbb{Q}$. Prove that Q is an equivalence relation. Can you describe its equivalence classes?

(Hints on page 271. Solutions on page 442.)

Chapter 16

Partitions

Exercise 104 (Counting)**.** *The goal of this exercise is to help you practice the definition of partition. Please try to answer it without referring back to the reference text; take a look at the hints if necessary.*

(a) Recall the definition of a **partition**.

(b) One common way of representing data is via a *pie chart*. Explain how a pie chart corresponds to a *partition* of the pie.

(c) How many partitions of $\emptyset$ are there?

(d) How many partitions of $\{1\}$ are there? What about $\{1, 2\}$?

(e) How many partitions of $\{1, 2, 3\}$ are there?

(Hints on page 272. Solutions on page 443.)

Exercise 105 (Find the Partitions)**.** *The goal of this exercise is to help you practice the definition of set partition. You can review Exercise 104 for the definition of a partition, but try to recall it first without consulting your answer!*

For each set A_i and each collection Ω_j, determine whether Ω_j is a partition of A_i. If not, explain why not.

(a) Let $A_1 = \{1, 2, 3, 4, 5, 6\}$

  (i) $\Omega_1 = \{\{1, 2\}, \{2, 3, 4\}, \{4, 5, 6\}\}$

  (ii) $\Omega_2 = \{\{1\}, \{2, 3, 6\}, \{4\}, \{5\}\}$

  (iii) $\Omega_3 = \{\{2, 4, 6\}, \{1, 3, 5\}\}$

  (iv) $\Omega_4 = \{\{1, 4, 5\}, \{2, 6\}\}$

  (v) $\Omega_5 = \{\{1, 2, 3, 4\}, \{5, 6\}, \{\}\}$

  (vi) $\Omega_6 = \{\{1, 2, 3, 4, 5, 6\}\}$

(b) Let $A_2 = \mathbb{Z}$

  (i) Ω_7 is the set containing the set of even integers and the set of odd integers.

  (ii) Ω_8 is the set containing the set of positive integers and the set of negative integers.

  (iii) Ω_9 is the set containing the set of integers strictly less than -100, the set of integers with absolute value less than or equal to 100, and the set of integers strictly greater than 100.

  (iv) Ω_{10} is the set containing the set of integers not divisible by 3, the set of even integers, and the set of integers that have a remainder of 3 when divided by 6.

(Hints on page 273. Solutions on page 445.)

Exercise 106 (Find the Partitions II)**.** *Here are slightly more challenging partition-spotting exercises. If you still feel unsure about the definition, it is worthwhile to take the time to think carefully about the pie-chart analogy in Exercise 104. You are also encouraged to create your own mnemonics!*

For each set A_i and each collection Ω_j, determine whether Ω_j is a partition of A_i. If not, explain why not.

(a) $A_1 = \mathbb{Z} \times \mathbb{Z}$

- (i) Ω_1 is the set containing the set of pairs (x, y) where x or y is odd, the set of pairs (x, y) where x is even, and the set of pairs (x, y) where y is even.
- (ii) Ω_2 is the set containing the set of pairs (x, y) where both x and y are odd, the set of pairs (x, y) where exactly one of x and y is odd, and the set of pairs (x, y) where both x and y are even.
- (iii) Ω_3 is the set containing the set of pairs (x, y) where x is positive, the set of pairs (x, y) where y is positive, and the set of pairs (x, y) where both x and y are negative.
- (iv) Ω_4 is the set containing the set of pairs (x, y) where $x > 0$ and $y > 0$, the set of pairs (x, y) where $x \leq 0$ and $y > 0$, and the set of pairs (x, y) where $x \leq 0$ and $y \leq 0$.
- (v) Ω_5 is the set containing the set of pairs (x, y) where $x \neq 0$ and $y \neq 0$, the set of pairs (x, y) where $x = 0$ and $y \neq 0$, and the set of pairs (x, y) where $x \neq 0$ and $y = 0$.

(b) Let $A_2 = \mathbb{R}$

- (a) $\Omega_6 = \{\{x \in \mathbb{R} | x < 0\}, \{0\}, \{x \in \mathbb{R} | x > 0\}\}$.
- (b) $\Omega_7 = \{$the set of irrational numbers, the set or rational numbers$\}$.
- (c) Ω_8 contains the sets of intervals $[k, k+1]$, where $k \in \mathbb{Z}$.
- (d) Ω_9 contains the sets of intervals $(k, k+1)$, where $k \in \mathbb{Z}$.
- (e) Ω_{10} contains the sets of intervals $(k, k+1]$, where $k \in \mathbb{Z}$.

(Hints on page 274. Solutions on page 446.)

Exercise 107 (Constructing Partitions)**.** *Among the infinitely many different partitions of* $\mathbb{N}$, *you are asked to find some examples satisfying specific requirements. This exercise (a variation on Exercise 7.56 from the reference text) will help sharpen your understanding of partitions (and ultimately, equivalence relations).*

(a) Find a partition of $\mathbb{N}$ that consists of 3 blocks, where 2 blocks contain a finite number of elements and the third block contains an infinite number of elements.

(b) Find a partition of $\mathbb{N}$ that contains an infinite number of blocks.

(c) Find a partition of $\mathbb{N}$ that consists of 3 blocks, where each block contains an infinite number of elements.

(d) Our definition of a partition requires a collection of subsets to satisfy three conditions. For each of these conditions, find a collection of subsets of $\mathbb{N}$ that fails this condition but satisfies the other two.

(Hints on page 275. Solutions on page 448.)

Exercise 108 (Relations from subsets)**.** *Even though the definition of partitions and equivalence relations look completely different, these concepts are two sides of the same coin. In this exercise you will explore the bridge connecting these two concepts, the process of associating a relation to a collection of subsets.*

Recall (Definition 7.62, p. 94 of the recommended reading) that given a collection Ω of subsets of A the **associated relation** R_Ω is defined by $aR_\Omega b$ if and only if there exists some $X \in \Omega$ such that $a, b \in X$.
Let $A = \{0, 1, 2, 3, 4, 5\}$. For each collection Ω_i below:

i. specify the corresponding relations R_{Ω_i} by listing the ordered pairs in the relation or by drawing the digraph corresponding to the relation;

ii. determine whether Ω_i is a partition.

iii. determine whether R_{Ω_i} is an equivalence relation and if so what are its equivalence classes.

(a) $\Omega_1 = \{\{0\}, \{1, 2\}, \{3, 4, 5\}\}$

(b) $\Omega_2 = \{\{0\}, \{1, 2\}, \{3, 4\}\}$

(c) $\Omega_3 = \{\{0, 1\}, \{2, 3\}, \{4, 5\}\}$

(d) $\Omega_4 = \{\{0, 1, 2\}, \{3, 4, 5\}\}$

(e) $\Omega_5 = \{\{0, 1\}, \{1, 2, 3\}, \{3, 4, 5\}\}$

(f) $\Omega_6 = \{\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}\}$

(Hints on page 276. Solutions on page 449.)

Exercise 109 (Relations and Partitions)**.** *This is "capstone exercise", where you are asked to prove several theorems from §7.3 of the recommended reading. This is an excellent exercise whether or not you've already proved these! You can check your understanding by proving or reproving these results.*

Recall (Definition 7.62, p. 94 of the recommended reading) that given a collection Ω of subsets of A the **associated relation** R_Ω is defined by $aR_\Omega b$ if and only if there exists some $X \in \Omega$ such that $a, b \in X$.

(a) Prove that R_Ω is always symmetric.

(b) Prove that R_Ω is reflexive if and only if Ω covers A.

(c) Prove that R_Ω is transitive if the sets in Ω are pairwise disjoint.

(d) Find an example where the sets in Ω are not pairwise disjoint but R_Ω is transitive.

(e) Conclude that if Ω is a partition of A then R_Ω is an equivalence relation on A.

(f) Suppose Ω is a partition of A, so that R_Ω is an equivalence relation; what are its equivalence classes?

(g) Suppose R_Ω is an equivalence relation on A. Prove that the equivalence classes form a partition of A.

(Hints on page 277. Solutions on page 452.)

Exercise 110 (Relations and Partitions II)**.** *In the previous exercise we've seen that every partition gives rise to an equivalence relation. We now prove that every equivalence relation gives rise to a partition (Theorem 7.59 in the recommended reading). These two exercises together demonstrate our claim that equivalence relation and partitions are two sides of the same coin—you can specify each by specifying the other.*

Suppose R is an equivalence relation on A. Prove that the equivalence classes form a partition of A.

(Hints on page 278. Solutions on page 454.)

Exercise 111 (Refinements)**.** *The idea of a "refinement" helps us compare how different partitions organize information. This shows up in diverse areas from data analysis to probability. Start by making sense of the definition in your own words before proceeding with the exercises.*

Let A be a set and let Ω_1, Ω_2 be two partitions of A. We say that Ω_1 is a **refinement** of Ω_2 if

$$\forall X \in \Omega_1 \, \exists Y \in \Omega_2 \, (X \subseteq Y).$$

(a) Let $A = \{1,2,3,4,5,6\}$, $\Omega_1 = \{\{1,2\},\{3\},\{4\},\{5,6\}\}$, and $\Omega_2 = \{\{1,2,3\},\{4,5,6\}\}$. Is Ω_1 a refinement of Ω_2? Justify your answer.

(b) Let $A = \{1,2,3,4,5,6\}$ and $\Omega_2 = \{\{1,2,3\},\{4,5,6\}\}$. Give an example of a partition Ω of A (different from Ω_1) that is a refinement of Ω_2.

(c) Let P_1, P_2, P_3 be partitions of a set B, and suppose that P_1 is a refinement of P_2, and that P_2 is a refinement of P_3. Prove or give a counterexample: P_1 is a refinement of P_3.

(d) Each partition of a set C corresponds to an equivalence relation on C, where two elements are equivalent if they lie in the same block (see Exercise 109). How does the fact that Q_1 is a refinement of Q_2 translate into a relationship between the equivalence relations R_{Q_1} and R_{Q_2} corresponding to Q_1 and Q_2, respectively? Explain your answer.

(Hints on page 279. Solutions on page 455.)

Chapter 17

Representatives

Exercise 112 (Representatives)**.** *This handout focuses on representatives of equivalence classes and when it is possible to define operations on equivalence classes in terms of representatives. This exercise both specializes and extends Theorem 7.42 from the recommended reading; test your understanding by trying to solve it without referring back to the text.*

Let R be an equivalence relation on A. Prove that for any $a, b \in A$ the following are equivalent:

(a) $[a] = [b]$;

(b) $a \in [b]$;

(c) aRb.

(Hints on page 280. Solutions on page 456.)

Exercise 113 (Operations)**.** *We would like to define operations on equivalence classes in terms of operations on their representatives; but we have to do so carefully, it is possible for the definition to not make sense!*

Consider the $\equiv_{10}$ equivalence relation on $\mathbb{Z}$; that is, for every $a, b \in \mathbb{Z}$, we have $a \equiv_{10} b$ if and only if $10|(b-a)$ (equivalently, if and only if a, b have the same last digit—see Exercise 6 of the Equivalence Relations Handout).

Suppose $\boxplus_i$ is a binary operation on $\mathbb{Z}$, and define $\oplus_i$ on the equivalence classes $\mathbb{Z}/\equiv_{10}$ by

$$[a] \oplus_i [b] = [a \boxplus_i b].$$

In each case, determine (with proof) whether $\oplus_i$ is well-defined. Recall that this means that if $[a] = [a']$ and $[b] = [b']$ then

$$[a] \oplus_i [b] = [a'] \oplus_i [b'].$$

(When adding equals to equals we must get the same result; otherwise $\oplus_i$ is not even an operation on $\mathbb{Z}/\equiv_{10}$.)[1]

(a) $a \boxplus_1 b = a$. (Then $b \boxplus_1 a = b$, so that in general $a \boxplus_1 b \neq b \boxplus_1 a$; this is an example of a non-commutative operation—see Exercise 114 below.)

(b) $a \boxplus_2 b$ is 0 if $a + b$ is even, and 1 if $a + b$ is odd.

(c) $a \boxplus_3 b$ is the remainder of $a + b$ when divided by 3.

(d) $a \boxplus_4 b = \min\{a, b\}$.

(e) $a \boxplus_5 b = 2a + 3b$.

(Hints on page 281. Solutions on page 457.)

[1] Binary operations on A can be thought of as functions $A \times A \to A$, so when we prove that an operation is well-defined we are actually checking that we have a legitimate function; the process here is more streamlined because (by defining everything in terms of representatives) we are guaranteed that we have at least a relation. Don't worry if this comment is confusing, we'll explore functions in depth in the next few handouts.

Exercise 114 (Properties)**.** *The advantage of defining operations on equivalence classes in terms of operations on their representatives is that all of the properties of the operation (on the representatives) are* inherited *(by the operation on the equivalence classes). This exercise generalizes Theorems 7.92 and 7.93 from the recommended reading.*

Let X be an arbitrary set equipped with a binary operation $\boxplus$ (this means that for $x, y \in X$ the symbol $x \boxplus y$ is again an element of X). Let R be an equivalence relation on X such that the operation $\oplus$ on the set of equivalence classes given by

$$[a] \oplus [b] = [a \boxplus b].$$

is well-defined.

(a) Suppose $\boxplus$ is associative. That is, for every $x, y, z \in X$ we have

$$(x \boxplus y) \boxplus z = x \boxplus (y \boxplus z).$$

Prove that $\oplus$ is associative. That is, prove that for equivalence classes $A, B, C \in X/R$ we have

$$(A \oplus B) \oplus C = A \oplus (B \oplus C).$$

(b) Suppose $\boxplus$ is commutative. That is, for every $x, y, z \in X$ we have

$$x \boxplus y = y \boxplus x.$$

Prove that $\oplus$ is commutative. That is, prove that for equivalence classes $A, B \in X/R$ we have

$$A \oplus B = B \oplus A.$$

(c) Suppose that $o \in X$ is an *identity element*. That is, for every $x \in X$ we have

$$o \boxplus x = x.$$

Prove that $\oplus$ also has an identity element; what is it?

(d) Suppose that every $x \in X$ has an *inverse*. That is, an element $y \in X$ such that the $\boxplus$-sum $x \boxplus y$ is the identity element of $\boxplus$:

$$x \boxplus y = o.$$

Prove that every equivalence $A \in X/R$ class also has an inverse (that is, an equivalence class $B \in X/R$ such that the $\oplus$-sum $A \oplus B$ is the identity element of $\oplus$—see part (c) above).

(e) What parts of your proofs use the fact that $\oplus$ is well-defined?

(Hints on page 282. Solutions on page 458.)

Note to the reader. *The next two exercises, Exercise 115 and Exercise 116, may appear long and abstract at first glance, but do not be intimidated! They are "capstone exercises" for the entirety of Chapter 7.*

Take your time to work through these exercises carefully, stopping to think and to make sure you understand the role of each part of the question, as well as the solution. By the end, you'd achieve something remarkable: not only would you cement your understanding of equivalence relations, partitions, and equivalence classes; but you'd discover new appreciation for such familiar objects as integers and fractions. The way mathematicians think of and construct these objects is very different from how they are first introduced.

The construction via equivalence classes is extremely powerful and ubiquitous throughout modern mathematics. The integers are equivalence classes of naturals, rationals are equivalence classes of integers, and reals are equivalence classes of rationals.

This continues throughout modern science: quantum mechanics is formalized in terms of geometric spaces whose "points" are (complex) functions, but for the "distance" between points to be well-defined, one must work with equivalence classes of functions.

Exercise 115 (The Integers)**.** *When we first encounter the integers, they look very similar to the natural numbers except for a possible "minus sign in front". There are many ways to rigorously define the integers, but the method presented here has become canonical because it leads to powerful generalizations.* On the set $Z = (\mathbb{N} \cup \{0\}) \times \mathbb{N}$ define a relation $\sim$ by

$$(a, b) \sim (c, d) \iff a + d = b + c.$$

(a) Prove that $\sim$ is an equivalence relation.

(b) Prove that $[(n, 0)]$ for $n \in \mathbb{N} \cup \{0\}$ and $[(0, n)]$ for $n \in \mathbb{N}$ determine a complete system of representatives: that is, every $(a, b) \in Z$ belongs to one of these equivalence classes and no two of these equivalence classes are the same.

The previous part allows us to *define* the integers as the set of equivalence classes of $\sim$. The idea is to *identify* $n \in \mathbb{N} \cup \{0\}$ with the equivalence class $[(n, 0)]$ and to *define* the symbol $-n$ as the equivalence class $[(0, n)]$.

(c) Define the operation $\boxplus$ on equivalence classes by

$$[(a, b)] \boxplus [(c, d)] = [(a + c, b + d)].$$

Prove that this operation is well-defined.

(d) We can think of the integers $\mathbb{Z}$ as an extension of $\mathbb{N} \cup \{0\}$ by finding "a copy" of $\mathbb{N} \cup \{0\}$ inside Z. We claimed that the idea is to identify $n \in \mathbb{N} \cup \{0\}$ with the equivalence class $[(n, 0)]$. The operation we have just defined on Z works well with this identification because it respects addition of natural numbers. Explain in your own words what this means.

Why should we "extend" the natural numbers via this abstract construction?

(e) We now get "a copy" of $\mathbb{N}$ "with a minus sign" by declaring that the meaning of the symbol $-n$ is the equivalence class $[(0, n)]$. Prove that $n + (-n) = 0$ by interpreting this as an operation on equivalence classes.

Instead of writing $x + (-y)$ we also write in shorthand $x - y$.

(f) Use our identification of numbers and operations as equivalence classes to prove that $5 - 2 = 3$ and that $2 - 5 = -3$.

(g) Prove that $[(a, b)]$ is identified with $n \in \mathbb{N} \cup \{0\}$ if and only if $[(b, a)]$ is identified with $-n$.

(h) Therefore, we define $\boxminus$ on Z by

$$\boxminus[(a, b)] = [(b, a)].$$

Prove that this is well-defined.

(i) Define

$$[(a, b)] \boxminus [(c, d)] = [(a, b)] \boxplus (\boxminus[(c, d)]) .$$

What is this operation in terms of representatives? That is, find e, f such that $[(a, b)] \boxminus [(c, d)] = [(e, f)]$. Is this a well-defined operation on Z? Check your definition in terms of representatives by proving again that $2 - 5 = -3$.

(Hints on page 283. Solutions on page 460.)

Exercise 116 (The Rationals)**.** *The construction of $\mathbb{Z}$ from Exercise 115 may have seemed unnecessarily abstract, but the same idea is crucial in the construction of the rational numbers.*

When we start learning about fractions, we want the symbol $\frac{1}{2}$ to be equal to the symbols $\frac{2}{4}, \frac{3}{6}, \frac{-4}{-8}, \dots$. In fact, the symbol $\frac{1}{2}$ stands for infinitely many other symbols. This is exactly the role of equivalence relations!

Let $Q = \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$. Define a relation $\sim$ on Q by

$$(a, b) \sim (c, d) \text{ if and only if } ad = bc.$$

The idea is that (a, b) stands for the symbol $\frac{a}{b}$.

(a) Prove that $\sim$ is an equivalence relation on Q.

We now *define* the symbol $\frac{a}{b}$ (with $b \neq 0$) as the equivalence class $[(a, b)]$. We want the rational numbers Q to extend the integers $\mathbb{Z}$. Our goal is to identify $z \in \mathbb{Z}$ with $\frac{z}{1}$, which is to say $[(z, 1)]$.

We want to be able to perform the same arithmetic operations of $\mathbb{Z}$ in Q. How should multiplication be defined? The simplest way is to multiply corresponding parts of our symbols $\frac{a}{b} \otimes \frac{c}{d} = \frac{ac}{bd}$.

(b) Define $\otimes$ on Q by $[(a, b)] \otimes [(c, d)] = [(ac, bd)]$. Prove that $\otimes$ is well-defined.

(c) Prove that the operation we have just defined on Q works well with the identification of $\mathbb{Z}$ matching $z \in \mathbb{Z}$ to $[(z, 1)]$.

(d) Use our definition of the symbol $\frac{a}{b}$ to prove that for any $z \in \mathbb{Z} \setminus \{0\}$ we have $z \times \frac{1}{z} = 1$.

What about addition? When we first learn about fractions, one naturally wants to try the simple definition of adding numerators and denominators $\frac{a}{b} \boxplus \frac{c}{d} = \frac{a+c}{b+d}$. However, it doesn't make sense to define addition like that.

(e) Define $\boxplus$ on Q by $[(a, b)] \boxplus [(c, d)] = [(a + c, b + d)]$. Prove that $\boxplus$ is *not* well-defined.

We must resort to the more complicated definition via "common denominator" $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$.

(f) Define $\oplus$ on Q by $[(a, b)] \oplus [(c, d)] = [(ad + bc, bd)]$. Prove that $\oplus$ is well-defined.

(g) Prove that the operation we have just defined on Q works well with the identification of $\mathbb{Z}$ matching $z \in \mathbb{Z}$ to $[(z, 1)]$.

What's the point of extending $\mathbb{Z}$ to Q? Just like we extended $\mathbb{N}$ to $\mathbb{Z}$ in order to define a global subtraction operation, the point of fractions (from the perspective of algebra) is to define a global division operation.

(h) Define the operation $\div$ on $Q \setminus \{0\}$ by $\div[(a, b)] = [(b, a)]$, for $a \neq 0$. Prove that $\div$ is well-defined.

(i) We now define, for $c \neq 0$,

$$[(a, b)] \div [(c, d)] = [(a, b)] \otimes (\div[(c, d)]) .$$

Is this a well-defined operation?

(j) Use our definition of the symbol $\frac{a}{b}$ to prove that for any $a, b \in \mathbb{Z}$ with $b \neq 0$ we have $a \div b = \frac{a}{b}$. More generally, prove that (for $s, t, u \neq 0$) $\frac{r}{s} \div \frac{t}{u} = \frac{ru}{st}$.

(Hints on page 285. Solutions on page 463.)

Chapter 18

Introduction to Functions

Exercise 117 (Non Functions)**.** *It's easy to recognize functions that work as expected, but noticing when a relation fails to be a function takes deeper understanding. Exploring non-examples will help you see why each part of the definition matters.*

Determine why each of the following is *not* a function.

(a) $f : \{a, b, c, d\} \to \{1, 2, 3\}, f = \{(a, 1), (b, 2), (c, 3)\}$.

(b) $f : \{a, b, c, d\} \to \{1, 2, 3\}, f = \{(a, 1), (b, 2), (c, 3), (a, 3), (b, 1), (d, 3)\}$.

(c) $f : \mathbb{R} \to \mathbb{R}, f(x) = \frac{1}{x}$.

(d) $f : \mathbb{N} \to \mathbb{N}, f(x) = \sqrt{x}$.

(e) $f : \mathbb{R} \to \mathbb{R}, f(x) = \sqrt{x}$.

(f) $f \subseteq \mathbb{R} \times \mathbb{R}$ is defined by $(x, y) \in f$ if and only if $x = |y|$ (the absolute value of y).

(Hints on page 287. Solutions on page 466.)

Exercise 118 (Function Construction). *A great way to test your understanding of a definition is to build your own examples and counterexamples. Constructing functions and non-functions from scratch will help you internalize what the rule "every input has exactly one output" really means.*

(a) List all the relations from $A = \{1\}$ to $B = \{a, b\}$. Indicate which ones are also functions.

(b) List all the functions from $C = \{1, 2\}$ to $D = \{a, b, c\}$.

(c) Give two examples of functions from $E = \{1, 2, 3\}$ to $F = \{a, b, c\}$ and two examples of relations from E to F that are not functions.

(d) How many different functions are there from E to F? Explain your reasoning.

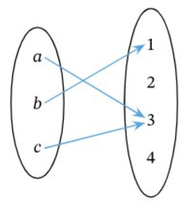
(e) Suppose M, N are finite sets with m, n elements (respectively). How many different functions are there from M to N? What about from N to M?

(Hints on page 288. Solutions on page 467.)

Exercise 119 (Is This a Function?). *Functions can be described in many ways; through formulas, graphs, tables, or sets of pairs. Working through these examples will help you recognize functions and non-functions no matter how they're described.*

For each of the following, determine whether the given rule or relation defines a function from the stated domain to the stated codomain. If it is a function, explain why. If not, explain why not.

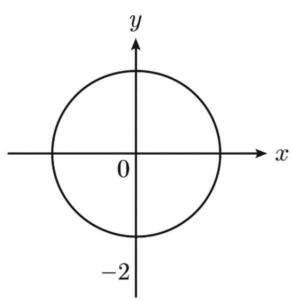
(a) The relation represented by the following digraph:



(b) $R \subseteq \{1,2,3,4\} \times \mathbb{R}, R = \{(1,\pi);(3,1);(4,\ln(5))\}$.

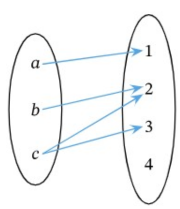
(c) $f : \mathbb{Z} \to \mathbb{Z},\ f(x) = x^2 + 1$.

(d) The relation $[-1,1] \times \mathbb{R}$ whose ordered pairs are the points in the the following graph:



(e) $S = \{(x,y) \in \mathbb{N} \times \mathbb{N} : y = x + 1\}$.

(f) The relation represented by the following digraph:



(g) $g : \mathbb{Z} \to \mathbb{Z},\ g(n) =$ the number of digits in (the usual decimal representation of) n.

(h) The relation $\{1,2,3,4\} \times \{1,2,3,4\}$ with pairs (x,y) represented by the following table:

x	1	2	3	4
y	1	1	2	3

(Hints on page 289. Solutions on page 468.)

Exercise 120 (Domain and Range)**.** *In this exercise you will practice identifying a function's domain and computing its range.*

Each of the following functions has $\mathbb{R}$ as its codomain. In each case, determine the domain and range.

(a) The function that assigns to each nonnegative integer its last digit (in the usual decimal representation).

(b) The function that assigns to each letter in English its position in the alphabet.

(c) The function that assigns to each pair of positive integers the maximum of these integers.

(d) The function that assigns to each finite sequence of 0's and 1's, the number of times the symbol 0 appears in the string.

(e) The function that assigns to each real number its square.

(Hints on page 290. Solutions on page 469.)

Exercise 121 (Codomain versus Range)**.** *A function's codomain tells us where outputs* could *live, but the range shows what values* actually *appear. This exercise examines how the choice of codomain may affect the properties of a function.*

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2 + 1$; and $g : \mathbb{R} \to [1, \infty)$ be defined by $g(x) = x^2 + 1$.

(a) State the domain, codomain, and range of f. Is the codomain of f equal to the range of f? If not, give an example of a value that is in one of these sets and not the other.

(b) State the domain, codomain, and range of g. Is the codomain of g equal to the range of g? If not, give an example of a value that is in one of these sets and not the other.

(c) Explain why it is possible for two functions to have exactly the same rule, but still be different functions.

(Hints on page 291. Solutions on page 470.)

Exercise 122 (Special Functions)**.** *Some of the simplest functions help clarify what a function is: a rule that consistently assigns a single output to each input. The inclusion, identity, and constant maps all satisfy this rule in different ways, highlighting how the domain and codomain shape the function's behavior.*

Let $A = \{1, 2, 3, 4\}$, and $B = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

(a) Explain why the inclusion map $\iota : A \to B, \iota(x) = x$ is a function from A to B. What would go wrong if we tried to define $\iota : B \to A$ using the same rule?

(b) How does the identity map $i_A : A \to A$ differ from $\iota : A \to B$?

(c) Define the constant function $c : A \to B$ by $c(x) = 6$ for every $x \in A$. State the domain, codomain, and range of c.

(d) Suppose we try to define $c : A \to \mathbb{N}$ by $c(x) = -1$. Would c be well-defined? Why, or why not?

(Hints on page 292. Solutions on page 471.)

Exercise 123 (Piecewise-Defined Functions)**.** *A piecewise definition can describe complex behaviors compactly, but only if each input value produces exactly one output. When the pieces overlap or leave gaps, the rule may fail to define a function.*

Consider the following rule for $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} x^2 & \text{if } 0 < x \leq 1 \\ 2x + 1 & \text{if } x \geq 1 \\ 3x - 4 & \text{if } x < 0. \end{cases}$$

(a) Compute $f(1)$ and $f(0)$ according to each applicable piece. Show explicitly how this leads to ambiguity.

(b) Revise the conditions in the definition of f so that it becomes well-defined. Carefully explain how the changes you've made result in a well-defined function.

(c) Suppose $h : A \to B$ is a piecewise-defined function

$$h(x) = \begin{cases} b_1 & \text{if } x \in A_1; \\ b_2 & \text{if } x \in A_2; \\ b_3 & \text{if } x \in A_3. \end{cases}$$

(Where $b_1, b_2, b_3 \in B$.) What conditions on A_1, A_2, A_3 (or b_1, b_2, b_3) do we need to check to ensure that h is well-defined?

(Hints on page 293. Solutions on page 472.)

Exercise 124 (The Ceiling and Floor Functions)**.** *Some functions are defined using everyday rounding ideas. The ceiling and floor functions assign to each real number the nearest integer above or below it, illustrating how precise wording ensures a function is well-defined.*

For any real number x, the ceiling function is defined by

$$\lceil x \rceil = \text{ the least integer greater than or equal to } x.$$

And the floor function is defined by

$$\lfloor x \rfloor = \text{ the greatest integer less than or equal to } x.$$

(a) Compute the following values:

$$\lceil 2.1 \rceil, \quad \lfloor 2.1 \rfloor, \quad \lceil -2.1 \rceil, \quad \lfloor -2.1 \rfloor.$$

(b) Describe in words what each of these functions does to a number.

(c) Draw the graphs of the ceiling and floor functions for the inputs x in the interval $[-3, 3]$.

(d) What would go wrong if we define the ceiling function by

$$\lceil x \rceil = \text{ an integer greater than or equal to } x?$$

(e) Prove that $r - \lfloor r \rfloor \in [0, 1)$.

(f) Prove that any real number r has a *unique* representation as a sum $r = n + \theta$, where $n \in \mathbb{Z}$ is an integer and $\theta \in [0, 1)$ lies on the unit interval.

(Hints on page 294. Solutions on page 473.)

Exercise 125 (Functions and Equivalence Relations)**.** *When defining operations on equivalence classes one has to be careful to check they are well-defined (cf. Handout on Representatives); this is true more generally for functions defined via representatives, as operations are just a special case of functions.*

In each case, determine whether the function is well-defined. If it is, provide a proof; if not, a counterexample.

(a) Let $\equiv_{10}$ be the equivalence relation on $\mathbb{Z}$ defined by $x \equiv_{10} y$ if and only if $10|y-x$. Let $f : \mathbb{Z}/\equiv_{10} \to \mathbb{Z}$ be defined by $f([x]) = x$.

(b) Let $f : \mathbb{Z}/\equiv_{10} \to \mathbb{Z}/\equiv 10$ be defined by $f([x]) = [x]$.

(c) Let $f : \mathbb{Z}/\equiv_{10} \to \mathbb{Z}$ be defined by $f([x])$ is the last digit in the usual decimal representation of x.

(d) Let $\sim$ be the relation on $\mathbb{R}\times\mathbb{R}$ defined by $(x, y) \sim (x', y')$ if and only if $x = x'$. Let $f : (\mathbb{R}\times\mathbb{R})/\sim \to \mathbb{R}$ be defined by $f([(x, y)]) = x$.

(e) Let $\sim$ be the relation on $\mathbb{R}\times\mathbb{R}$ defined by $(x, y) \sim (x', y')$ if and only if $x = x'$. Let $f : (\mathbb{R}\times\mathbb{R})/\sim \to \mathbb{R}$ be defined by $f([(x, y)]) = y$.

(f) Let Z be the equivalence relation on $\mathbb{R}$ defined by xZy if and only if $y - x \in \mathbb{Z}$ (cf. Exercise 7 from the Equivalence Relations handout). Let $f : \mathbb{R}/Z \to \mathbb{R}$ be defined by $f[x] = x - \lfloor x \rfloor$ (where $\lfloor x \rfloor$ is the floor function—see Exercise 124).

(Hints on page 295. Solutions on page 475.)

Exercise 126 (Some Properties of the Ceiling and Floor Functions)**.** *The is a **bonus** exercise examining the subtle behaviour of the ceiling and floor functions when combined with addition or scaling. It is a great opportunity to practice your proof writing skills, one of the main focuses of our course!*

Refer to Exercise 124 for the definitions of the ceiling and floor functions $\lceil x \rceil$ and $\lfloor x \rfloor$. Part (f) may be especially useful with proving some of the following properties.

(a) For a real number x, prove that

$$\lfloor 2x \rfloor = \lfloor x \rfloor + \left\lfloor x + \frac{1}{2} \right\rfloor.$$

(b) Prove or give a counterexample: for any $x, y \in \mathbb{R}$,

$$\lceil x + y \rceil = \lceil x \rceil + \lceil y \rceil.$$

(c) Prove or give a counterexample: for any $x, y \in \mathbb{R}$,

$$\lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor.$$

(d) Prove that for any $x \in \mathbb{R}$,

$$\left\lceil \frac{x}{2} \right\rceil + \left\lfloor \frac{x}{2} \right\rfloor = \begin{cases} \lfloor x \rfloor & \text{if } \lfloor x \rfloor \text{ is odd;} \\ \lceil x \rceil & \text{if } \lfloor x \rfloor \text{ is even.} \end{cases}$$

(Hints on page 296. Solutions on page 476.)

Chapter 19

Injective and Surjective Functions

Exercise 127 (Basic definitions)**.** *New mathematical definitions should be investigated from many different angles; one should compare and contrast similar sounding statements, construct examples and non-examples, find special or extremal cases, discover equivalent formulations, and so on. This exercise will help you practice the definitions of injectivity, surjectivity, and bijectivity.*

Let $f : X \to Y$ be a function. For each of the following conditions, determine whether the condition guarantees f to be injective, surjective, bijective, or none of the above. If none of the above, give an example where the conditions fail.

(a) For every $x \in X$ there is some $y \in Y$ such that $f(x) = y$.

(b) For every $y \in Y$ there is some $x \in X$ such that $f(x) = y$.

(c) $X = Y$.

(d) For every $x \in X$ there is exactly one $y \in Y$ such that $f(x) = y$.

(e) For every $y \in Y$ there is exactly one $x \in X$ such that $f(x) = y$.

(f) For every two elements $x \neq x' \in X$, we have $f(x) \neq f(x')$.

(g) For every two elements $y \neq y' \in Y$ there are $x \neq x' \in X$ such that $f(x) = y$ and $f(x') = y'$.

(h) For every two elements $x, x' \in X$, if $f(x) = f(x')$ then $x = x'$.

(i) For every two elements $x, x' \in X$, $f(x) = f(x')$ if and only if $x = x'$.

(Hints on page 297. Solutions on page 478.)

Exercise 128 (Classifying functions)**.** *Determining whether a function is injective, surjective, or bijective plays an important role in many proofs. This exercise will help you identify these properties in different functions.*

Determine (with proof) which of the following functions is injective, surjective, or bijective.

(a) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 3x + 2$.

(b) $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x^2$.

(c) $h : [0, \infty) \to [0, \infty)$ defined by $h(x) = x^2$.

(d) $k : \mathbb{R} \to [0, \infty)$ defined by $k(x) = x^2$.

(e) $p : \mathbb{N} \to \mathbb{N}$ defined by $p(n) = n + 1$.

(Hints on page 298. Solutions on page 479.)

Exercise 129 (Piecewise-defined Function)**.** *In many cases a function is defined in a piecewise fashion. In the previous handout we have examined conditions under which such a procedure is well-defined; we now turn to the question of which properties it preserves.*

Let A, B, C, D be sets with $A \cap C = \emptyset$. Let $f : A \to B$ and $g : C \to D$ be functions, and define $h : A \cup C \to B \cup D$ by

$$h(x) = \begin{cases} f(x) & \text{if } x \in A; \\ g(x) & \text{if } x \in C. \end{cases}$$

(a) Explain why h is well-defined.

(b) Suppose f, g are injective, does it follow that h is injective? Prove or provide a counterexample.

(c) Suppose f, g are surjective, does it follow that h is surjective? Prove or provide a counterexample.

(d) Suppose $B \cap D = \emptyset$ and f, g are bijective, does it follow that h is bijective? Prove or provide a counterexample.

(Hints on page 299. Solutions on page 480.)

Exercise 130 (Finite sets). *Injections, surjections, and bijections play a crucial role in Set Theory. From the perspective of Set Theory, the main property of sets are their sizes (formally,* cardinality*), and maps between sets can give information about their relative sizes. Injections, surjections, and bijections are used as the* definitions *of size comparison between infinite sets! The second half of this question is a bit challenging; consult the hints if you need to, but only after giving them an honest try!*

Let $m, n \in \mathbb{N}$ and set $A = \{1, 2, \dots, n\}$ and $B = \{1, 2, \dots, m\}$.

(a) Suppose $n \leq m$. Construct an injection $A \to B$.

(b) Suppose $n \geq m$. Construct a surjection $A \to B$.

(c) Suppose $n = m$. Construct a bijection $A \to B$.

(d) Suppose $f : A \to B$ is a surjection; prove that $n \geq m$.

(e) Suppose $g : A \to B$ is an injection; prove that $n \leq m$.

(f) Suppose $h : A \to B$ is a bijection; prove that $n = m$.

(g) Conclude that

- $n \leq m$ if and only if there exists an injection $A \to B$;
- $n \geq m$ if and only if there exists a surjection $A \to B$;
- $n = m$ if and only if there exists a bijection $A \to B$.

(Hints on page 300. Solutions on page 481.)

Exercise 131 (Constructing a bijection)**.** *Constructing bijections between different sets is a fundamental tool in almost all branches of mathematics, from algebra to combinatorics to topology and so on. This exercises challenges you to construct a bijection; even when you know a bijection must exist, it is not always trivial to find!*

Fix some $m, n \in \mathbb{N}$ and let $A = \{1, 2, \ldots, m\}$ and $B = \{1, 2, \ldots, n\}$. When we studied the Cartesian product, we noted that $|A \times B| = mn$. Therefore, by Exercise 130, there must be a bijection $A \times B \to \{1, 2, 3, \ldots, mn\}$. Our goal is to construct such a bijection!

We know that $|A \times B| = mn$ because we can arrange the elements of $A \times B$ in an $m \times n$ table.

$$\begin{array}{ccccc}
(1,1) & (1,2) & (1,3) & \cdots & (1,n) \\
(2,1) & (2,2) & (2,3) & \cdots & (2,n) \\
(3,1) & (3,2) & (3,3) & \cdots & (3,n) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
(m,1) & (m,2) & (m,3) & \cdots & (m,n)
\end{array}$$

To prove there are mn elements in this table, we can count them one by one. There are many ways of doing so, but let's imagine we start from the top left and continue in the same order as reading a page of English. That is, $(1,1)$ is the first element, then $(1,2)$ is the second, $(1,3)$ the third, until we reach the end of the line with $(1,n)$ the n-th element. Then we move on to the next line, with $(2,1)$ being the $n+1$ element, $(2,2)$ the $n+2$ element and so forth.

(a) Fix some $1 \leq r \leq m$. If we continue counting according to the procedure above, what number would be the first element in row r?

(b) Fix some $1 \leq k \leq n$. If we continue counting according to the procedure above, what number would be the element (r,k)?

(c) Construct a map $\Phi : A \times B$ and $\{1, 2, 3, \ldots, mn\}$ which matches the element $(r,k) \in A \times B$ to the its "count".

(d) Prove that Φ is injective.

(e) Prove that Φ is surjective.

(Hints on page 301. Solutions on page 483.)

Exercise 132 (Set difference)**.** *Injective and surjective maps interact with other set and functional operations. In addition to serving as a great practice problem, the full significance of this exercise will become more apparent when we study the difference between the behaviours of the image and preimage, in a future handout.*

Let $f : X \to Y$ be a function. Recall that for any $Z \subseteq X$, the notation $f[Z]$ stands for the image of Z under f, i.e.

$$f[Z] = \{f(z) \,:\, z \in Z\}\,.$$

(a) Prove that for any $A, B \subseteq X$,

$$f[A] \setminus f[B] \subseteq f[A \setminus B].$$

(b) Give an example (specify X, Y, f, A, B) where the $f[A] \setminus f[B] \neq f[A \setminus B]$.

(c) Prove that f is injective if and only if for any $A, B \subseteq X$,

$$f[A \setminus B] = f[A] \setminus f[B].$$

(Hints on page 302. Solutions on page 484.)

Exercise 133 (Cantor's Theorem)**.** *One of the most fundamental and surprising results in Set Theory is Cantor's Theorem. It is used to prove the* non-existence *of a surjective map. In Set Theory it implies the existence of different "sizes" of infinities. The idea is a refined version of Russell's Paradox (see §3.2 of the Recommended Text) and appears in yet another version in Computer Science as part of Turing's proof of the fundamental limitation of computation.*

For any finite set X, say with $|X| = n$, we have seen that $|\mathcal{P}(X)| = 2^n$. Since $2^n > n$, Exercise 130 implies there is no surjection $X \to \mathcal{P}(X)$. Our goal is to extend this conclusion to arbitrary (possibly infinite) sets.

Let X be an arbitrary set and assume for contradiction that there is a surjection $f : X \to \mathcal{P}(X)$. Note that for any $x \in X$ the value $f(x) \in \mathcal{P}(X)$ is a set, a subset of X in fact. Let us define

$$Y := \{x \in X \,:\, x \notin f(x)\}\,.$$

Then Y is a subset of X (possibly empty). Argue that there is no $x \in X$ with $f(x) = Y$, contradicting the assumption that f is surjective. Conclude that there is no surjective map $X \to \mathcal{P}(X)$, for any set X.

(Hints on page 303. Solutions on page 485.)

Chapter 20

Compositions and Inverse Functions

Exercise 134 (Composition of functions)**.** *This exercise will help you practice forming and evaluating compositions in a variety of settings.*

Find the following compositions of functions, if they exist; don't forget to specify the domain and codomain.

(a) Let $f : \mathbb{Z} \to \mathbb{Z}, f(x) = x + 2$, and $g : \mathbb{Z} \to \mathbb{Z}, g(x) = 3x^3$. Find $f \circ g$ and $g \circ f$.

(b) Let $f : \mathbb{Z}_6 \to \mathbb{Z}_6, f(x) = x^2 \pmod 6$ (or $[x^2]_6$ in the notation of the recommended reading), and $g : \mathbb{Z}_6 \to \mathbb{Z}_6, g(x) = 3x \pmod 6$ (or $[3x]_6$). Compute $f \circ g$ and $g \circ f$.

(c) Let $f : \mathbb{Z} \to \mathbb{R}, g(x) = x - 3$, and let $f : (0, \infty) \to \mathbb{R}, f(x) = \sqrt{x}$. Find $f \circ g$ and $g \circ f$.

(d) Let $A = \{a, b\}$, $B = \{1, 2, 3\}$, and $C = \{x, y\}$. Define $f : B \to C$ by $f(1) = x, f(2) = y, f(3) = y$; define $g : A \to B$ by $g(a) = 2$ and $g(b) = 3$. Find $f \circ g$ and $g \circ f$.

(e) Let $f : \mathbb{Z}_5 \to \mathbb{Z}_2, f([x]_5) = [x + 1]_2$ and $g : \mathbb{Z}_2 \to \mathbb{Z}_2, g([x]_2) = [x + 1]_2$. Find $f \circ g$ and $g \circ f$.

(Hints on page 304. Solutions on page 486.)

Exercise 135 (Order of composition)**.** *Commuting operators play a key role in almost all branches of mathematics and theoretical physics. This exercise focuses on (1-dimensional, real) affine operators.*

Let $f, g : \mathbb{R} \to \mathbb{R}$ be the affine functions $f(x) = ax + b$ and $g(x) = cx + d$ (where $a, b, c, d \in \mathbb{R}$ are fixed constants).

(a) Give an example (by finding real numbers a, b, c, d) such that $g \circ f \neq f \circ g$.

(b) Are there values of a, b, c, d such that $g \circ f = f \circ g$? If so, characterize those values; if not, explain why not.

(c) Is there an affine function h which commutes with *every* affine function? That is, $h \circ f = f \circ h$ no matter the value of $a, b \in \mathbb{R}$. If so, find all such functions; if not, explain why not.

(Hints on page 305. Solutions on page 487.)

Exercise 136 (Compositions and injectivity)**.** *This exercise will help you build intuition for how information is "lost" or "preserved" as functions are composed; it collects together aspects of several Problems and Theorems from the recommended text.*

Let $f : X \to Y$ and $g : Y \to Z$ be functions, so that the composition $g \circ f : X \to Z$ is well-defined.

(a) Prove Theorem 8.60 from the recommended reading: If f, g are injective, then so is their composition $g \circ f$.

(b) Show that the converse fails: if $g \circ f$ is injective it does not follow that both f, g are injective.

(c) Is it possible for both f, g to be non-injective and for $g \circ f$ to be injective?

(d) Suppose $Z = X$ and $g \circ f = i_X$ is the identity function on X. What must be true about f, g?

(Hints on page 306. Solutions on page 488.)

Exercise 137 (Compositions and surjectivity)**.** *This exercise complements Exercise 136 by examining how surjectivity behaves under composition; it collects aspects of several Problems and Theorems from the recommended text.*

Let $f : X \to Y$ and $g : Y \to Z$ be functions, so that the composition $g \circ f : X \to Z$ is well-defined.

(a) Prove Theorem 8.61 from the recommended reading: If f, g are surjective, then so is their composition $g \circ f$.

(b) Show that the converse fails: if $g \circ f$ is surjective it does not follow that both f, g are surjective.

(c) Is it possible for both f, g to be non-surjective and for $g \circ f$ to be surjective?

(d) Suppose $Z = X$ and $g \circ f = i_X$ is the identity function on X. What must be true about f, g?

(Hints on page 307. Solutions on page 489.)

Exercise 138 (Left- and right-inverses)**.** *This exercise explores (in a practical, hands-on fashion) the connection between the functional characteristic of injectivity/surjectivity and the algebraic characteristic of composition: existence of left- and right-inverses.*

(a) Complete the following table

	injective	surjective	has a left-inverse	has a right-inverse
f_1				
f_2				
f_3				
f_4				
f_5				

where

$f_1 : \{1, 2\} \to \{a, b, c\}$ is given by $f_1(1) = a$, $f_1(2) = b$.
$f_2 : \{1, 2, 3\} \to \{a, b\}$ is given by $f_2(1) = a$, $f_2(2) = a$, $f_2(3) = b$.
$f_3 : \mathbb{R} \to \mathbb{R}$ is given by $f_3(x) = 2x + 1$.
$f_4 : \mathbb{R} \to \mathbb{R}$ is given by $f_4(x) = x^2$.
$f_5 : \mathbb{Z}_6 \to \mathbb{Z}_6$ is given by $f_5(x) = 3x \pmod 6$.

(b) For each function that has a left-inverse or a right-inverse, compute them.

(c) Explain how to minimally modify the domain and/or codomain so that each function above has a left-inverse and/or a right-inverse; or explain why such a modification is not possible.

(Hints on page 308. Solutions on page 490.)

Exercise 139 (Inverse relation). *Any relation can be inverted to create another relation; but if we start with a function the result of the inversion may fail to be a function. Each way the inverted relation may fail imposes a constraint on the function we started with.*

Recall that given a relation $R \subseteq A \times B$ the *inverse relation* $R^{-1} \subseteq B \times A$ is defined by $(b, a) \in R^{-1} \iff (a, b) \in R$.

(a) Prove that $(R^{-1})^{-1} = R$ (as mentioned in the recommended reading, this is a generalization of Theorem 8.81).

In the Introduction to Functions handout we identified two ways a relation R may fail to be a function. Suppose R is a *function*.

(b) Suppose R^{-1} fails to be a function because some $b \in B$ does not have any $a \in A$ such that $(b, a) \in R^{-1}$. What does this tell us about the function R?

(c) Formulate (and prove!) a necessary and sufficient condition on the function R so that every $b \in B$ has at least one $a \in A$ for which $(b, a) \in R^{-1}$.

(d) Suppose R^{-1} fails to be a function because some $b \in B$ has two $a, a' \in A$ such that $(b, a); (b, a') \in R^{-1}$. What does this tell us about the function R?

(e) Formulate (and prove!) a necessary and sufficient condition on the function R so that every $b \in B$ has at most one $a \in A$ for which $(b, a) \in R^{-1}$.

(f) Formulate (and prove!) a necessary and sufficient condition on the function R so that R^{-1} is a function.

(Hints on page 309. Solutions on page 491.)

Exercise 140 (Two-sided inverse)**.** *When a function has a one-sided inverse (either a left-inverse or a right-inverse) there are usually many choices; but if a function has both a left- and a right-inverse, they must coincide! In this exercises you are asked to prove this remarkable fact.*

(a) Let us revisit the function $f_1 : \{1, 2\} \to \{a, b, c\}$ (given by $f_1(1) = a, f_1(2) = b$) from Exercise 138. Construct two distinct functions g_1, g_1' each of which is a left-inverse for f_1.

(b) Let us revisit the function $f_2 : \{1, 2, 3\} \to \{a, b\}$ (given by $f_2(1) = a, f_2(2) = a, f_2(3) = b$) from Exercise 138. Construct two distinct functions g_2, g_2' each of which is a right-inverse for f_2.

(c) Let $f : X \to Y$ be a function which has a left-inverse $g : Y \to X$ and a right-inverse $h : Y \to X$. Prove that $g = h$.

(d) Conclude that if a function f has both a left-inverse and a right-inverse, it has a unique left inverse, a unique right inverse, a unique two-sided inverse, and all of these coincide. We can therefore denote this unique inverse by the special symbol f^{-1} with no ambiguity.

(e) Suppose $f : X \to Y$ and $g : Y \to Z$ are functions for which two-sided inverses exist. Prove that $g \circ f$ also has a two-sided inverse and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$. (This is Theorem 8.82 from the recommended reading.)

(Hints on page 310. Solutions on page 492.)

Exercise 141 (**Bonus:** Cantor–Schröder–Bernstein Theorem)**.** *In the Injective and Surjective Functions handout we've seen (Exercise 4) that for finite sets, if $f : X \to Y$ is injective than $|X| \leq |Y|$; if $g : Y \to X$ is injective then $|X| \geq |Y|$. It follows that $|X| = |Y|$, so there should be a bijection between them! The goal of this exercise is to prove this result for* all *sets, not only finite sets. This is the celebrated Cantor–Schröder–Bernstein Theorem and the elegant proof below is due to König. This exercise is significantly longer and more challenging than usual, but by this point in the course you are ready! Please take it slowly and be patient while solving it.*

Suppose $f : X \to Y$ is an injective function and $g : Y \to X$ is an injective function. Our goal is to build a bijective function $h : X \to Y$; to do so, we need to assign a value for each $x \in X$.

(a) Suppose $\phi : A \to B$ is an injective function. Prove that for any $b \in B$ the set $\phi^{-1}(\{b\})$ contains either 0 elements or 1 element.

Our idea is to partition the domain and codomain into three blocks, and match those blocks together. The cleverness of the proof is in the definition of the blocks, which is achieved via sequences of function ϕ for the domain and ψ for the codomain.

We define two sequences of functions as follows: $\phi_0 = g$, $\psi_0 = f$ and for each $n \in \mathbb{N}$,

$$\phi_n = \begin{cases} \phi_{n-1} \circ f & \text{if } n \text{ is odd;} \\ \phi_{n-1} \circ g & \text{if } n \text{ is even.} \end{cases} \qquad \psi_n = \begin{cases} \psi_{n-1} \circ g & \text{if } n \text{ is odd;} \\ \psi_{n-1} \circ f & \text{if } n \text{ is even.} \end{cases}$$

(b) Use mathematical induction to prove that ϕ_n, ψ_n are well-defined *injective* functions for each $n \in \mathbb{N}$.

(c) Prove that for each $n \in \mathbb{N}$ and each $x \in X$, the set $\phi_n^{-1}(\{x\})$ contains either 0 elements or 1 element. Similarly, for each $n \in \mathbb{N}$ and each $y \in Y$, the set $\psi_n^{-1}(\{y\})$ contains either 0 elements or 1 element.

(d) Prove that for every $n \in \mathbb{N}$ we have $f \circ \phi_{n-1} = \psi_n$ and $g \circ \psi_{n-1} = \phi_n$.

(e) Prove that

$$\forall x \in X \ \forall n \in \mathbb{N} \ (\psi_n^{-1}(\{f(x)\}) = \phi_{n-1}^{-1}(\{x\})).$$

What is the analogous statement for $y \in Y$?

We now partition the domain X and the codomain Y into three sets:

$$X_{no} := \{x \in X \ : \ \text{for every } n \in \mathbb{Z}_{\geq 0},\ \phi_n^{-1}(\{x\}) \neq \emptyset\}\,;$$
$$X_{odd} := \{x \in X \ : \text{the smallest } n \in \mathbb{Z}_{\geq 0} \text{ such that } \phi_n^{-1}(\{x\}) = \emptyset \text{ is odd}\}\,;$$
$$X_{even} := \{x \in X \ : \text{the smallest } n \in \mathbb{Z}_{\geq 0} \text{ such that } \phi_n^{-1}(\{x\}) = \emptyset \text{ is even}\}\,.$$

The sets $Y_{no}, Y_{even}, Y_{odd}$ are defined in an analogous fashion (with ψ replacing ϕ). For example,

$$Y_{no} := \{y \in Y \ : \ \text{for every } n \in \mathbb{Z}_{\geq 0},\ \psi_n^{-1}(\{y\}) \neq \emptyset\}\,.$$

(f) Prove that $f(X_{no}) = Y_{no}$. Moreover, that $f' : X_{no} \to Y_{no}$ given by $f'(x) = f(x)$ is a bijection.

(g) Prove that $f(X_{even}) = Y_{odd}$. Moreover, that $f'' : X_{even} \to Y_{odd}$ given by $f''(x) = f(x)$ is a bijection.

(h) Prove that $g(Y_{even}) = X_{odd}$. Moreover, that $g' : X_{odd} \to X_{even}$ given by $g'(x) = g^{-1}(x)$ is a bijection.

(i) Construct a bijection $h : X \to Y$.

(Hints on page 311. Solutions on page 494.)

Chapter 21

Images and Preimages of Functions

Exercise 142 (Notation)**.** *It sometimes happen that the same notation or word is used with different meanings, which one has to distinguish based on context. At the beginning, one has to pay special attention to guard against confusion, but with some practice the meaning becomes apparent.*

Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$. Which of the following are defined, and what do they mean? If it is defined, compute it; if not, explain why not.

(a) $f^{-1}(3)$

(b) $f^{-1}(\{3\})$

(c) $f^{-1}(\{-3\})$

(d) $f^{-1}(x)$

(e) $f^{-1}([0, 1])$

(Hints on page 313. Solutions on page 497.)

Exercise 143 (Images and Preimages). *This exercise will help you practice finding images and preimages of sets under different functions, building intuition for how functions transform subsets of their domain and codomain.*

(a) Let $S = \{-1, 0, 2, 4, 7\}$. Find $f(S)$ if

(i) $f : \mathbb{R} \to \mathbb{R}$ is the constant function $f(x) = 1$.

(ii) $f : \mathbb{R} \to \mathbb{R}$ is the linear function $f(x) = 2x + 1$.

(iii) $f : S \to \mathbb{Z}$ is the inclusion function $\iota : S \to \mathbb{Z}$.

(iv) $f : \mathbb{R} \to \mathbb{R}$ is the function $f(x) = \lceil \frac{x}{5} \rceil$. (Here, $\lceil \cdot \rceil$ is the ceiling function; see the Introduction to Functions handout.)

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be the doubling function $f(x) = 2x$. What is $f(S)$ if

(i) $S = \{-2, -1, 0, \frac{1}{2}, \frac{5}{6}, \pi\}$.

(ii) $S = \mathbb{N}$.

(iii) $S = \mathbb{Z}$.

(iv) $S = \mathbb{R}$.

(c) Let $f : \mathbb{R} \to \mathbb{R}$ be the absolute value function $f(x) = |x|$. Find each of the following preimages:

(i) $f^{-1}(\{4\})$.

(ii) $f^{-1}([2, 8])$.

(iii) $f^{-1}(\mathbb{Z})$.

(iv) $f^{-1}((-\infty, 0])$.

(Hints on page 314. Solutions on page 498.)

Exercise 144 (Preimages and Complements)**.** *This problem will help you practice the definitions and logical structure of preimages, set operations, and how they interact. See also Problem 8.89 from the recommended reading.*

Let $f : X \to Y$ be a function and $S \subseteq Y$ a subset of the codomain.

(a) Write out the definitions of the sets $f^{-1}(S)$ and S^c symbolically, in terms of logical statements about elements.

(b) Using these definitions, prove that

$$f^{-1}(S^c) = (f^{-1}(S))^c.$$

(c) Express this equality in words; how do complements behave under preimages?

(Hints on page 315. Solutions on page 499.)

Exercise 145 (Images of Intersections). *This exercise explores an important connection between properties of functions and operations on sets. See also Problem 8.89 from the recommended reading.*
Let $f : X \to Y$ be some arbitrary function and $A, B \subseteq X$ arbitrary subsets of the domain.

(a) Show that,
$$f(A \cap B) \subseteq f(A) \cap f(B).$$

(b) Give an example where equality holds and an example where equality fails.

(c) Find a sufficient condition on f that guarantees equality:
$$f(A \cap B) = f(A) \cap f(B).$$

(d) Prove that your condition is necessary: if $f(A \cap B) = f(A) \cap f(B)$ for all $A, B \subseteq X$ then the function must be ...?

(Hints on page 316. Solutions on page 500.)

Exercise 146 (The Characteristic Function)**.** *Characteristic functions provide a bridge between set theory and algebra. By expressing membership in algebraic form, we can reinterpret familiar set operations as arithmetic formulas. This problem will help you connect ideas of images, preimages, and logical structure with algebraic representations of sets.*

Fix some "universal set" U. Any subset $S \subseteq U$ defines a *characteristic function* $\chi_S : U \to \{0,1\}$ by

$$\chi_S(x) = \begin{cases} 1 & \text{if } x \in S; \\ 0 & \text{if } x \notin S. \end{cases}$$

(a) What is the range of $\chi_\emptyset$, of χ_U? What about χ_S for $\emptyset \subsetneq S \subsetneq U$?

(b) Fix some $S \subseteq U$. For each subset of $\{0,1\}$, find its *preimage* under χ_S That is, determine

$$\chi_S^{-1}(\{1\}), \quad \chi_S^{-1}(\{0\}), \quad \chi_S^{-1}(\{0,1\}), \quad \chi_S^{-1}(\emptyset).$$

(c) Prove that for any $S \subseteq X$ we have $(\chi_S)^2 = \chi_S$.

Show that for all $A, B \subseteq U$, we have

(d) $\chi_{A\cap B} = \chi_A \cdot \chi_B$.

(e) $\chi_{A\cup B} = \chi_A + \chi_B - \chi_A \cdot \chi_B$.

(f) $\chi_{A^c} = 1 - \chi_A$.

(g) Using the results above, express the characteristic function of the *symmetric difference* $A \triangle B = (A \cup B) \setminus (A \cap B)$ in terms of χ_A and χ_B.

(Hints on page 317. Solutions on page 502.)

Exercise 147 (The Characteristic Function of $\mathbb{Z}$)**.** *This problem explores how the characteristic function of the integers, $\chi_{\mathbb{Z}} : \mathbb{R} \to \{0,1\}$, can be described using elementary real functions such as the floor and ceiling. This exercise connects the discrete nature of $\mathbb{Z}$ with real-valued function of a real parameter.*

Recall that for any real number x we have (see the Introduction to Functions handout):

$$\lfloor x \rfloor = \text{the greatest integer less than or equal to } x;$$
$$\lceil x \rceil = \text{the least integer greater than or equal to } x.$$

(a) Suppose $x \in \mathbb{Z}$, what is the value of $\lfloor x \rfloor$? What about $\lceil x \rceil$?

(b) Prove that $\lfloor x \rfloor \leq \lceil x \rceil$.

(c) Referring to the previous part, when does equality hold? Formulate (and prove) sufficient and necessary conditions for equality.

(d) Referring to part (b) above, prove that if equality does not hold, then $\lfloor x \rfloor$ and $\lceil x \rceil$ are consecutive integers.

(e) Express the characteristic function of the integers $\chi_{\mathbb{Z}} : \mathbb{R} \to \{0,1\}$ using only the ceiling and floor functions (and algebra). Recall that (cf. Exercise 146)

$$\chi_{\mathbb{Z}}(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Z}; \\ 0 & \text{if } x \notin \mathbb{Z}. \end{cases}$$

(Hints on page 319. Solutions on page 504.)

Exercise 148 (Functions, Preimages, and Partitions)**.** *The recommended reading explores in detail how each function gives rise to an equivalence relations. We know that equivalence relations and partitions are two sides of the same coin, so it follows that a function naturally partitions its domain. This problem will help you make this connection explicit.*

Let $f : X \to Y$ be a function.

(a) Prove that distinct values have disjoint preimages: if $y_1 \neq y_2 \in Y$, then $f^{-1}(\{y_1\}) \cap f^{-1}(\{y_2\}) = \emptyset$.

(b) Show that

$$X = \bigcup_{y \in Y} f^{-1}(\{y\}).$$

(c) Does the collection of preimages

$$\{\, f^{-1}(\{y\}) \mid y \in Y \,\}$$

form a *partition* of X? If so, prove it; if not, formulate a related correct statement and prove it.

(d) Suppose $f : \mathbb{R} \to \mathbb{Z}$ is the floor function $f(x) = \lfloor x \rfloor$ (see the Introduction to Functions handout). Describe in detail the collection of preimages.

(Hints on page 320. Solutions on page 505.)

Part II

Hints

Hint for Exercise 1 (Vocabulary)**.**

Below are some examples of **correct** usage:

$$A := \{1, 2, 3, 4\} \qquad 1 \in A \qquad 2|4 \qquad 3 \in \mathbb{N} \qquad -5 \in \mathbb{Z} \qquad \sqrt{2} \in \mathbb{R}$$

(Exercise on page 11.)

Hint for Exercise 2 (Parity).

Here is the definition of "even" and "odd" from the text:

> **Definition 2.1.** An integer n is **even** if $n = 2k$ for some $k \in \mathbb{Z}$. An integer n is **odd** if $n = 2k + 1$ for some $k \in Z$.

(a) To prove that 246 is even, we need to replace n everywhere in the definition with 246.

> The integer 246 is even if $246 = 2k$ for some $k \in \mathbb{Z}$.

(b) To prove that 101 is *not even* we must show that it is *impossible* to satisfy the definition. That is, we must prove that there is no $k \in Z$ such that $n = 2k + 1$.

(Exercise on page 12.)

Hint for Exercise 3 (Squaring)**.**

Here is the definition of "even" and "odd" from the text:

> **Definition 2.1.** An integer n is **even** if $n = 2k$ for some $k \in \mathbb{Z}$. An integer n is **odd** if $n = 2k + 1$ for some $k \in Z$.

We are *given* that n is an even integer. This means that $n = 2k$ for some $k \in \mathbb{Z}$. We want to *prove* that n^2 is an even integer, so we need to find some $j \in \mathbb{Z}$ such that $n^2 = 2j$.

(Exercise on page 13.)

Hint for Exercise 4 (Divisibility).

(a) It may help to use different variable names in the two definitions:

> **Definition 2.1.** An integer n is **even** if $n = 2k$ for some $k \in \mathbb{Z}$. An integer n is **odd** if $n = 2k + 1$ for some $k \in Z$.
>
> **Definition 2.5.** Given $a, b \in \mathbb{Z}$, we say that a **divides** b, written $a|b$, if there exists $r \in \mathbb{Z}$ such that $b = ar$. If $a|b$, we may also say that b is **divisible by** a or that a is a **factor** of b.

We are *given* that n is even, so that $n = 2k$ for some $k \in \mathbb{Z}$. We want to *prove* that n is divisible by 2. What should be the values of a, b, r in Definition 2.5?

(b) What are we *given* and what are we trying to *prove*? What should be the values of the relevant parameters in Definition 2.1 and Definition 2.5?

(c) The two statements are logically different from each other!

(d) Would it be enough to show that $n \neq 2 \cdot 1$? What about that $n \neq 2 \cdot 1$ and $n \neq 2 \cdot 2$ and $n \neq 2 \cdot 3$?

(e) Recall a helpful assumption made in the text

> "For the remainder of this section, you may assume that every integer is either even or odd but never both."

(Exercise on page 14.)

Hint for Exercise 5 (Translation)**.**

Try substituting the propositions with symbols, one at a time, where it makes sense.

(Exercise on page 15.)

Hint for Exercise 6 (Truth Values)**.**

Try assigning truth values (that is, assuming that a proposition is true or assuming it is false) and figure out how the truth value of the compound propositions are affected.

(Exercise on page 16.)

Hint for Exercise 7 (Truth Tables)**.**

Start by recalling the name of each connective symbol:

- $\neg$ is the *negation* symbol;
- $\wedge$ is read as "and" (called the *conjunction* symbol);
- $\vee$ is read as "or" (called the *disjunction* symbol);
- $\implies$ is read as "if ... then ..." (called the *conditional* symbol or the *(material) implication* symbol);
- $\iff$ is read as "if and only if" (called the *biconditional* symbol or *bi-implication* symbol).

(Exercise on page 17.)

Hint for Exercise 8 (Arithmetic)**.**

Once you know how to express $\neg$ and $\wedge$, you can express the other connectives as well. DeMorgan's Laws and the different forms of implication would help!

(Exercise on page 18.)

Hint for Exercise 9 (Propositions)**.**

Consult the previous exercise about truth tables; try to "plug in" the appropriate truth values for each component to find the relevant row of the table.

(Exercise on page 19.)

Hint for Exercise 10 (Cards)**.**

(a) One useful way to start is to identify the connective, which will determine the compound proposition.

(b) Remember the analogy of an implication as a "contract".

(Exercise on page 20.)

Hint for Exercise 11 (Nested)**.**

Remember the truth table of $A \implies B$; there is a single row which evaluates to "false".

(Exercise on page 21.)

Hint for Exercise 12 (Words).

(a) The converse and contrapositive both have to do with changing the direction of implication, but the inverse does not.

(b) The descriptions all fall into two categories $A \implies B$ or $B \implies A$.

(Exercise on page 22.)

Hint for Exercise 13 (Equivalence)**.**

All of these equivalences are related to implication, its negation, and DeMorgan's Laws.

(Exercise on page 23.)

Hint for Exercise 14 (Complete Set of Connectives)**.**

In addition to DeMorgan's Laws, the following equivalences will prove useful (all of them can be verified directly from the truth-table):

- $A \implies B$ is logically equivalent to $(\neg A) \vee B$;
- $\neg(A \implies B)$ is logically equivalent to $A \wedge (\neg B)$ (this also follows from the previous equivalence using DeMorgan's Laws);
- A is logically equivalent to $\neg(\neg A)$.

(Exercise on page 24.)

Hint for Exercise 15 (Tautologies and Contradictions).

Remember the truth table of $A \implies B$; there is a single row which evaluates to "false". Exercise 11 may also be helpful.

(Exercise on page 25.)

Hint for Exercise 16 (Contrapositive Statements)**.**

Recall that the contrapositive of the implication $A \implies B$ is the statement $\neg B \implies \neg A$.

(Exercise on page 27.)

Hint for Exercise 17 (Direct Proofs)**.**

If you want to prove the implication $A \implies B$ by a direct proof, you begin by assuming A. Next you use definitions and known results to derive B.

For example, for part (a) you would assume that a and b are integers and that $a \mid b$. Then use the definition of $a \mid b$ to prove that for an arbitrary integer c, we have $a \mid bc$.

(Exercise on page 28.)

Hint for Exercise 18 (Contra- Proofs)**.**

Recall that the contrapositive of the implication $A \implies B$ is the statement $\neg B \implies \neg A$.

To prove a conditional statement $A \implies B$ by contraposition, you begin by assuming $\neg B$ and then use definitions and known results to conclude $\neg A$.

To prove a conditional statement $A \implies B$ by contradiction, you begin by assuming A, and for the sake of contradiction, assume $\neg B$, then use definitions and known results to reach a contradiction (prove that some statement P and its negation $\neg P$ must both be true). You can then conclude B.

(Exercise on page 29.)

Hint for Exercise 19 (Direct Proofs II)**.**

If you want to prove the implication $A \implies B$ by a direct proof, you begin by assuming A. Next you use definitions and known results to derive B.

The following familiar facts from algebra may be useful:

- The product of two positive numbers is positive.
- If $a > b$ and $b > c$, then $a > c$.
- $(a-b)(a+b) = a^2 - b^2$.

(Exercise on page 30.)

Hint for Exercise 20 (Contra- Proofs II)**.**

Recall that the contrapositive of the implication $A \implies B$ is the statement $\neg B \implies \neg A$.

To prove a conditional statement $A \implies B$ by contraposition, you begin by assuming $\neg B$ and then use definitions and known results to conclude $\neg A$.

To prove a conditional statement $A \implies B$ by contradiction, you begin by assuming A, and for the sake of contradiction, assume $\neg B$, then use definitions and known results to reach a contradiction (prove that some statement P and its negation $\neg P$ must both be true). You can then conclude B.

(Exercise on page 31.)

Hint for Exercise 21 (Prove or Disprove).

(a) Try proving the contrapositive, or using a parity table.

(b) Try splitting your proof into cases, when n is an odd integer and when n is an even integer.

(c) Consider $(m+n)^2$. What will its parity be? Can you relate it to m^2+n^2?

(d) This is a biconditional statement, so make sure you prove both directions.

(e) Try factoring n^2-1.

(Exercise on page 32.)

Hint for Exercise 22 (Proposition vs Predicate).

A **proposition** is a statement that is either true or false. Recall the example from the textbook:

> the sentence "All dogs have four legs" is a false proposition.

A **predicate** is a statement with one or more free variables. Recall the example from the textbook:

> the perfectly good sentence "$x = 1$" is not a proposition all by itself since we do not actually know what x is.

A variable such as x above is said to be **free**; in contrast, a variable is **bound** if it is inside the scope of a quantifier, as in

$$\forall x \in \mathbb{R}\,(x = 1).$$

(Exercise on page 33.)

Hint for Exercise 23 (Vocabulary of Quantifiers)**.**

Recall that $\forall x \in A, P(x)$ means that *every* element of A satisfies $P(x)$. $\exists x \in A, P(x)$ means that there is *at least one* $x \in A$ that satisfies $P(x)$.

(Exercise on page 34.)

Hint for Exercise 24 (Finite Universe of Discourse)**.**

Recall that $\forall x\, P(x)$ means that *every* element in the universe of discourse satisfies $P(x)$. In particular, $P(1)$ is true and so is $P(2)$ and so is... Can you construct a mathematical statement which encapsulates all the true statements implied by $\forall x\, P(x)$?

(Exercise on page 35.)

Hint for Exercise 25 (Changing the Universe of Discourse).

Take the first statement for example $\forall x\,(x^2 \geq 0)$. When the universe of discourse is $\mathbb{N}$, the statement asserts that the square of every natural number is nonnegative. When the universe of discourse is $\mathbb{Z}$, the statement asserts that the square of every integer is nonnegative. When the universe of discourse is $\mathbb{R}$, the statement asserts that the square of every real number is nonnegative. As you can see, each of these statements asserts something distinct, and its truth-value should be evaluated.

(Exercise on page 36.)

Hint for Exercise 26 (Translating Quantified Statements)**.**

The order of the quantifiers matters. For example when we say $\forall x\, \exists y\, P(x, y)$, we're saying that for each x there is a y, possibly dependent on x, such that $P(x, y)$ is satisfied. Whereas $\exists y\, \forall x\, P(x, y)$ says that there is at least one y that satisfies $P(x, y)$ no matter what the value of x is.

(Exercise on page 37.)

Hint for Exercise 27 (Evaluating Quantified Statements)**.**

Consider the first statement, for example, $\forall x \in \mathbb{R}\, \exists y \in \mathbb{R}\, (x + y = 0)$. The statement asserts that for every real number x there is some real number y such that $x + y = 0$. Is this true? If someone gives us an x, what kind of y can we choose? Is there an x for which no y can be found?

If the order of quantification is reversed, we get the statement $\exists y \in \mathbb{R}\, \forall x \in \mathbb{R}\, (x+y = 0)$. This statement asserts that there is one real number y such that no matter what x is chosen it will always be the case that $x+y = 0$. Is this statement true? What real number y can satisfy this property? If someone claims to have such a y, can we find an x that proves them wrong?

(Exercise on page 38.)

Hint for Exercise 28 (Exchanging Quantifiers)**.**

You may wish to consult the previous exercise, Exercise 27.

(Exercise on page 39.)

Hint for Exercise 29 (Set-Builder Notation).

Set-builder notation is when you describe a set S as follows

$$S = \{x \in A \mid P(x)\}$$

where $P(x)$ is a predicate statement involving x. To express a set in set-builder notation, you start by specifying A, the 'domain', or the 'space' the elements of the set come from, and you follow that by the conditions, $P(x)$, that need to be satisfied so that the element is a part of your set.

Unpack the definitions of the sets in set builder notation by observing where the elements come from, and what conditions they must satisfy.

(Exercise on page 41.)

Hint for Exercise 30 (Subsets).

We say that A is a subset of B, denoted $A \subseteq B$, if every element of A is also an element of B. More formally, if $(x \in A) \implies (x \in B)$. We say that A is a *proper* subset of B, denoted $A \subsetneq B$ if A is a subset of B and $A \neq B$.

How does this definition translate to the empty set? What elements does the empty set contain?

(Exercise on page 42.)

Hint for Exercise 31 (Set Equality)**.**

Recall that to show that two sets A and B are equal, $A = B$, you need to prove two things; that $A \subseteq B$ and that $B \subseteq A$.

(Exercise on page 43.)

Hint for Exercise 32 (Set Operations).

Recall that the **union** of the sets A and B in the universe U is

$$A \cup B := \{x \in U \mid x \in A \text{ or } x \in B\}.$$

And the **intersection** of the sets A and B in the universe U is

$$A \cap B := \{x \in U \mid x \in A \text{ and } x \in B\}.$$

The **set difference** of the sets A and B in the universe U is defined as

$$A \setminus B := \{x \in U \mid x \in A \text{ and } x \notin B\}.$$

While the **complement** of the set A in the universe U is defined as

$$A^c := U \setminus A = \{x \in U \mid x \notin A\}.$$

Additionally, we say that two sets A and B are disjoint, if they have an empty intersection; in other words, if $A \cap B = \emptyset$.

(Exercise on page 44.)

Hint for Exercise 33 (The Empty Set)**.**
Recall that the **union** of the sets A and B in the universe U is

$$A \cup B := \{x \in U \mid x \in A \text{ or } x \in B\}.$$

The **intersection** of the sets A and B in the universe U is

$$A \cap B := \{x \in U \mid x \in A \text{ and } x \in B\}.$$

And the **complement** of the set A in the universe U is

$$A^c := U \setminus A = \{x \in U \mid x \notin A\}.$$

How does this translate to the empty set? What elements are contained in the empty set?

(Exercise on page 45.)

Hint for Exercise 34 (Properties of Set Operations)**.**

(a) Recall that A is a subset of B, denoted $A \subseteq B$, if every element of A is also an element of B. More formally, if $(x \in A) \implies (x \in B)$. To prove the theorem, you can use a direct proof. Assume $A \subseteq B$ and $B \subseteq C$, and try to prove that $A \subseteq C$.

(b) Fix some sets A, B and C and work out your example step by step to find the elements of each set that appears in the Theorem.

(c) Recall the definition of the union of sets A and B,

$$A \cup B := \{x \in U \mid x \in A \text{ or } x \in B\}.$$

You can show that two sets are equal via double subset inclusion (proving that each set is a subset of the other). You can also show that two sets are equal by proving that they contain exactly the same elements. Think of how you can do this using set-builder notation.

(d) Recall the definition of the intersection of sets A and B,

$$A \cap B := \{x \in U \mid x \in A \text{ and } x \in B\}.$$

(e) Recall the definition of the complement of a set A in universe U,

$$A^c := U \setminus A = \{x \in U \mid x \notin A\}.$$

(Exercise on page 46.)

Hint for Exercise 35 (Subset Equivalences).

Notice that these are all biconditionals, so make sure you prove both directions for each biconditional.

Recall that A is a subset of B, denoted $A \subseteq B$, if every element of A is also an element of B. More formally, if $(x \in A) \implies (x \in B)$.

Also, the **union** of the sets A and B in the universe U is defined by

$$A \cup B := \{x \in U \mid x \in A \text{ or } x \in B\}.$$

And the **intersection** of the sets A and B in the universe U is defined by

$$A \cap B := \{x \in U \mid x \in A \text{ and } x \in B\}.$$

(Exercise on page 47.)

Hint for Exercise 36 (Set Equalitites)**.**

You can show that two sets are equal by double subset inclusion (proving that each set is a subset of the other). You can also show that two sets are equal by proving that they contain exactly the same elements. Think of how you can do this using set-builder notation. You can also use a series of set equivalences that you've already proven.

Remember that $A^c = U \setminus A$.

(Exercise on page 48.)

Hint for Exercise 37 (Union-Complement Form)**.**

You can use De Morgan's law, distribution of unions and intersections, the double complement property, and any other set equivalence that you have proven.

(Exercise on page 49.)

Hint for Exercise 38 (Symmetric Difference)**.**

Unpack the definition of $A \Delta B$ and compute each part separately. First find $A \setminus B$, then find $B \setminus A$. Finally find their union.

(Exercise on page 50.)

Hint for Exercise 39 (Power Set Definition)**.**

The power set of S is the set of subsets of S, i.e., sets that "share all their elements" with A. What are some subsets of A? For example, $\emptyset \subseteq A$ (the empty set "shares all its element" with A, it just has nothing to share!). We also have $\{1\} \subseteq A$ (because every element of $\{1\}$ is also an element of A). Can you find all the subsets of A?

(Exercise on page 51.)

Hint for Exercise 40 (Power Set Computation)**.**

Consider the set $B = \{a, b\}$ for example. This is very similar to the set $\{1, 2\}$ from the previous exercise (pretend that a means 1 and b means 2, for example). Can you list all the subsets of B? There should be four of them in total!

(Exercise on page 52.)

Hint for Exercise 41 (Power Set Cardinality)**.**

How many elements are in the power set of a set with one element, say $S = \{a\}$?
How many elements are in the power set of a set with two elements, say $S = \{a, b\}$?
How many elements are in the power set of a set with three elements, say $S = \{a, b, c\}$?
Do you notice a pattern?

(Exercise on page 53.)

Hint for Exercise 42 (Possible Power Sets)**.**

Recall that the power set of a set S must contain all possible subsets of S. In particular, there is one set that is definitely an element of every power set!

(Exercise on page 54.)

Hint for Exercise 43 (Power Set Closures)**.**

This is an exercise in "unpacking" definitions. Take part (a) for example. If $X, Y \in \mathcal{P}(S)$ then $X \subseteq S$ and $Y \subseteq S$. We want to prove that $X \cup Y \in \mathcal{P}(S)$, so can you show that $X \cup Y \subseteq S$?

(Exercise on page 55.)

Hint for Exercise 44 (Set Operations and Power Sets)**.**

Here are some helpful strategies for "true or false" problems. First, try a few different examples to get a feel for whether the statement is true or false. In the process, you might find a counterexample that proves that the statement is false!

Another strategy is to start by trying to prove the statement. If the statement is true, you might just be done! Otherwise, you will get stuck in your proof and you can try to make that sticking point into a counterexample.

Below we proceed to give a hint to part (d), you should first try it on your own by using the strategies above.

Since $A \cap B \subseteq A, B$, we can use part (c). For the other direction, suppose $X \in \mathcal{P}(A) \cap \mathcal{P}(B)$. This means that $X \in \mathcal{P}(A)$ and $X \in \mathcal{P}(B)$, so that $X \subseteq A$ and $X \subseteq B$. In other words, every element of X is both an element of A and an element of B. Can you show that $X \subseteq A \subseteq B$?

(Exercise on page 56.)

Hint for Exercise 45 (Finite Indices)**.**

Recall that for a finite set of indices, as in here, we can "unpack" the "big union" notation to "small union" notation:

$$\bigcup_{i=1}^{3} S_i = S_1 \cup S_2 \cup S_3.$$

(Exercise on page 57.)

Hint for Exercise 46 (Infinite Indices)**.**

Recall that for the set $\mathbb{N}$ of indices, we can "unpack" the "big union" notation to "small union" notation:

$$\bigcup_{i=1}^{\infty} S_i = S_1 \cup S_2 \cup S_3 \cup \cdots .$$

For proving the correctness of our computation, we must use the definition of the "big union":

$$\bigcup_{i\in\mathbb{N}} S_i := \{x \,:\, x \in S_i \text{ for some } i \in \mathbb{N}\}.$$

(Exercise on page 58.)

Hint for Exercise 47 (Uncountable Unions)**.**

Recall the definition of the "big union" symbol:

$$\bigcup_{r\in\mathbb{R}} S_r := \{x \,:\, x \in S_r \text{ for some } r \in \mathbb{R}\}\,.$$

It may also help in the problem-solving process to graph the relevant sets! (Exercise on page 59.)

Hint for Exercise 48 (Uncountable Unions Revisited)**.**

You may use the fact that if $x \in \mathbb{R}$ then both x^3 and $\sqrt[3]{x}$ are real numbers. (The latter fact, about the existence of a unique cubic root, is typically proved in a Single Variable Calculus or Introduction to Real Analysis course.) (Exercise on page 60.)

Hint for Exercise 49 (Uncountable Intersections).

Recall the definition of the "big intersection" symbol:

$$\bigcap_{r\in\mathbb{R}} S_r := \{x : x \in S_r \text{ for all } r \in \mathbb{R}\}.$$

Again, it may help to graphically represent some of the sets. (Exercise on page 61.)

Hint for Exercise 50 (Uncountable Intersections Revisited)**.**

Recall the definition of the "big intersection" symbol:

$$\bigcap_{r\in\mathbb{R}} S_r := \{x : x \in S_r \text{ for all } r \in \mathbb{R}\}.$$

Again, it may help to graphically represent some of the sets. (Exercise on page 62.)

Hint for Exercise 51 (Unions and Intersections I)**.**

It may help to sketch the first few even-indexed intervals I_2, I_4, I_6.

Also, note the indices in the definition of $J_k := \bigcap_{n=k}^{\infty} I_{2n}$. The first index is k not 1.

(Exercise on page 63.)

Hint for Exercise 52 (Unions and Intersections II)**.**

The proof by contradiction starts with assuming that $x \in J_k$, and for the sake of contradiction, that $x < 2$. Also recall that you proved the reverse subset inclusion $[\frac{1}{2k}, 2] \subseteq J_k$ in Exercise 51.

It may help to sketch the first few odd-indexed intervals I_1, I_3, I_5 and, separately, the first few even-indexed intervals I_2, I_4, I_6.

To prove that your expression for $\bigcup_{k=1}^{\infty} E_k$ is correct, use double subset inclusion. Recall that two sets A and B are equal if and only if $A \subseteq B$ AND $B \subseteq A$. So you'll need to prove both subset inclusions. If you're stuck, you can look at the solution to see the claim for what $\bigcup_{k=1}^{\infty} E_k$ is equal to, but try not to look at the proof before attempting it yourself.

(Exercise on page 64.)

Hint for Exercise 53 (Unions and Intersections III).

It may help to break up the problem. First, find with proof an expression for $\bigcup_{n=k}^{\infty} I_n$. This can be further broken up into cases, when k is even and when k is odd. Next, find the intersection of these sets,

$$\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} I_n.$$

(Exercise on page 65.)

Hint for Exercise 54 (Monotone Sequences)**.**

(a) Notice how each set is contained within the following set. Which set would be contained in every other set?

(b) Notice how each set contains the following set. Which set would contain all the other sets?

(c) Recall that the complement of a set is all the elements (in a specified universe) that do not belong to the set. Also, this is a biconditional statement so don't forget to prove both directions.

(d) You can use the results of parts (a) and (c), and recall that $(A^c)^c = A$.

(Exercise on page 66.)

Hint for Exercise 55 (Pairwise Disjoint).

- It is enough to show that if $\ell < n$ then $S_n \cap S_\ell = \emptyset$. (Do you see why?)
- Assume for contradiction $x \in S_n \cap S_\ell$. Write x as something plus $1/n$ and also as something plus $1/\ell$.
- For every two real numbers m, k exactly one of $m < k$, $m = k$, or $m > k$ holds; show that each case leads to a contradiction.

(Exercise on page 67.)

Hint for Exercise 56 (Limits).

(a) Try to break it up. If $x \in \liminf S_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} S_n$, then $x \in \bigcap_{n=k}^{\infty} S_n$ for some $k \in \mathbb{N}$. What does this mean? Can you translate this to $\exists B \in \mathbb{N}.\forall j \in \mathbb{N}.[(j \geq B) \implies (x \in S_j)]$? Don't forget you also need to prove the reverse subset inclusion.

(b) Again, break it up first. If $x \in \limsup S_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} S_n$, then $x \in \bigcup_{n=k}^{\infty} S_n$ for some $k \in \mathbb{N}$. What does this mean? Can you translate this to $\forall B \in \mathbb{N}.\exists j \in \mathbb{N}.[(j \geq B) \wedge (x \in S_j)]$? Don't forget you also need to prove the reverse subset inclusion

(c) What does it mean for x to be an element of $\liminf S_n$? How does that imply that x is an element of $\limsup S_n$?

(d) Assume $\{S_n\}_{n \in \mathbb{N}}$ is pairwise disjoint and try to figure out what $\liminf S_n$ is equal to, in particular, what is $\bigcap_{n=k}^{\infty} S_n$ for an arbitrary positive integer k?

Next, find $\limsup S_n$ using the definition. Try finding the intersections of the first few sets. For example, what is $\bigcap_{k=1}^{2} \bigcup_{n=k}^{\infty} S_n$?

Finally, confirm that you get the same set for $\liminf S_n$ and $\limsup S_n$.

(e) You can use all the results that you've proven in the previous parts. You can also use the results of Exercise 54

(f) You can use all the results that you've proven in the previous parts. You can also use the results of Exercise 54

(g) Let $S_1 = S_3 = S_5 = \cdots = S_o$, so that S_o denotes the common value of all the odd-indexed sets. And let $S_2 = S_4 = S_6 = \cdots = S_e$, so that S_e denotes the common value of all the even-indexed sets. Now think of what being an element of $\liminf S_n$ or of $\limsup S_n$ means and try to figure out expressions for them in this case (remember that in this part, we only have two sets to deal with, S_o and S_e).

(Exercise on page 68.)

Hint for Exercise 57 (Tuples vs. Sets)**.**

Sets are defined solely by their elements; tuples are defined by the order of their elements.

(Exercise on page 69.)

Hint for Exercise 58 (Computing Products).

Recall that $A \times B$ is the set of all tuples (a, b) where the first element is from A, i.e. $a \in A$, and the second element is from B, i.e $b \in B$.

(Exercise on page 70.)

Hint for Exercise 59 (Empty Products)**.**

Use the formal definition of the Cartesian product of sets:

$$A \times B = \{(a, b) \,:\, a \in A, b \in B\}.$$

It may help to proceed by contradiction for the first two statements and by contrapositive for the last one.

(Exercise on page 71.)

Hint for Exercise 60 (Properties of Cartesian Products).

For commutativity it may be useful to consult the previous exercises.

For associativity, suppose $a \in A$, $b \in B$, and $c \in C$. What is an example of an element of $A \times B$? How about $B \times C$? How about $(A \times B) \times C$? Can you show it is not possible for this element to be in $A \times (B \times C)$? What is the definition of this last set? It may help to write it in two stages, from the "outside in" as $S \times C$ where $S = A \times B$. (Exercise on page 72.)

Hint for Exercise 61 (Criteria for Commutativity).

Here are some hints for each part. If you read the hint for one part, try the next part by yourself before looking at the hint!

(a) To prove $A \times B \subseteq C \times D$ we start with an arbitrary element $(a, b) \in A \times B$ and we wish to show it must be an element of $C \times D$. We know that $a \in A$ and $b \in B$. We are given $A \subseteq C$ and $B \subseteq D$. Can you prove that $(a, b) \in C \times D$?

(b) We wish to prove that $A \subseteq C$ so we start with an arbitrary $a \in A$ and need to show it is an element of C. We are given that $A \times B \subseteq C \times D$; how can we use this data and how does it connect to our arbitrary $a \in A$? Try to form an ordered pair with a.

(c) Try to use the previous parts with a suitable definition of C, D.

(Exercise on page 73.)

Hint for Exercise 62 (Product Projections)**.**

Here are some hints for each part. If you read the hint for one part, try the next part by yourself before looking at the hint!

(a) Look at Exercise 58.

(b) Work out the edge cases $S = \emptyset$ and $S = A \times B$ first.

(c) Try a few geometric examples where $A = B = \{0, 1, 2, \ldots, 9\}$, since this is easy to draw on a plane.

(d) In the previous part, think what exactly went wrong in case you found a counterexample.

(Exercise on page 74.)

Hint for Exercise 63 (Visualizing Products).

Here are some hints for each part. If you read the hint for one part, try the next part by yourself before looking at the hint!

(a) Locate the corners of the region first. For example, the x-coordinate is all points a such that $0 < a < 1$. So first consider the points $a = 0, 1$ (even if these are not part of the set, they give you the boundary and a concrete idea of where to start). Can you work out the y-coordinates of the corners?

(b) Do the same as above, mark the origin $(0, 0)$ and first identify the boundaries of the region.

(c) What is $\mathbb{R} \times \{0\}$? What is $\mathbb{R} \times \{0, 1, 2\}$?

(Exercise on page 75.)

Hint for Exercise 64 (Distributivity of the Product I).

Look at Exercise 58 for a reminder on how to compute the product. The union, intersection and set difference were covered in previous sections as well. Before explicitly finding all the elements of a set, how many elements do you expect?

(Exercise on page 76.)

Hint for Exercise 65 (Distributivity of the Product II).

Here are some hints for each part. If you read the hint for one part, try the next part by yourself before looking at the hint!

(a) Try proving it by double-containment. If all the steps are logically reversible (that is, the converse also holds), then maybe you can find a more direct proof.

(b) How is this statement similar to the one above?

(c) Try double-containment.

(d) Try using what you learned in the first three parts.

(e) Ask yourself: is the fact that the given operation is on the right of the $\times$ operation essential to the idea of the proof? Try reproving these statements in your head.

(Exercise on page 77.)

Hint for Exercise 66 (Product and other Set Operations)**.**

Here are some hints for each part. If you read the hint for one part, try the next part by yourself before looking at the hint!

(a) Try proving it by double-containment.

(b) Attempt a proof by double-containment, and pay attention to the step where you get stuck.

(c) Based on your failed proof from above, one of the two containments always holds. What is missing from the other one?

(Exercise on page 78.)

Hint for Exercise 67 (Product and other Set Operations II).

Here are some hints for each part. If you read the hint for one part, try the next part by yourself before looking at the hint!

(a) Consider the examples in the previous two exercises.

(b) You can always expand the complements by their definition.

(Exercise on page 79.)

Hint for Exercise 68 (Distributivity Revisited)**.**

Since this is a generalization, you should carefully look at the proof of Exercise 65 and try to imitate the argument in this new general case. A good intermediate point would be to generalize the statements to three sets, instead of two.

(Exercise on page 80.)

Hint for Exercise 69 (Inductive reasoning).

(a) Plug in $n = 1$ into the universally quantified statement.

(b) Write down your explanations in words. Each statement will lead you to the next.

(c) Imagine climbing an infinite staircase.

(Exercise on page 81.)

Hint for Exercise 70 (Recap).

(a) Plug $n = 1, \ldots, 5$ into the formula for T_n.

(b) Recall that a predicate is a statement with a variable that for each value of the variable is either true or false. An equation involving n is either true or false for each value of n.

(c) Check both sides of the equation when you set $n = 1$.

(d) Write out what the statement would be when we plug in $n + 1$ in place of n.

(e) Assuming

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2},$$

we want to use this to prove $P(n + 1)$, to do this we compute

$$1 + 2 + 3 + \cdots + n + (n + 1).$$

(i) Fill in the blank

$$1 + 2 + 3 + \cdots + n + (n + 1) = \underbrace{\ldots\ldots\ldots\ldots\ldots}_{\text{inductive hypothesis}} + (n + 1).$$

(ii) Simplify the expression on the right. Show that it is equal to $\frac{(n+1)(n+2)}{2}$. Explain why this completes the inductive step.

(Exercise on page 82.)

Hint for Exercise 71 (Writing inductive proofs).

(a) The first odd number is 1, the second odd number is 3, the third odd number is 5, and so on. Moreover, the first square number is $1^2 = 1$, the second square number is $2^2 = 4$, the third square number is $3^2 = 9$, and so on.

(b) We are looking for an equation in terms of n. This is similar to what we did in Exercise 70.

(c) Plug in $n = 1$ into the predicate you've defined.

(d) Write out what the statement would be when we plug in $n + 1$ into your predicate in place of n.

(e) Assuming

$$1 + 3 + 5 + \cdots + 2n - 1 = n^2,$$

we want to use this to prove $P(n + 1)$, to do this we compute

$$1 + 3 + 5 + \cdots + (2n - 1) + (2n + 1).$$

(i) Fill in the blank

$$1 + 3 + 5 + \cdots + (2n - 1) + (2n + 1) = \underbrace{\ldots\ldots\ldots\ldots\ldots}_{\text{inductive hypothesis}} + (2n + 1).$$

(ii) Simplify the expression on the right. Show that it is equal to $(n + 1)^2$. Explain why this completes the inductive step.

(Exercise on page 83.)

Hint for Exercise 72 (Writing inductive proofs II)**.**

- Start by clearly defining $P(n)$.
- Verify the base case.
- Explain what is the inductive hypothesis.
- Use the inductive hypothesis to complete the proof of the inductive step $\forall n \in \mathbb{N}(P(n) \implies P(n+1))$.
- To prove the inductive step, fill in the blank

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) + (n+1)(n+2) = \underbrace{\ldots\ldots\ldots\ldots}_{\text{by inductive hypothesis}} + (n+1)(n+2).$$

  Simplify the expression on the right to get $\frac{(n+1)(n+2)(n+3)}{3}$.

(Exercise on page 84.)

Hint for Exercise 73 (False proof).

(a) Write out what the statement would be when we plug in $n+1$ into the predicate in place of n.

(b) Fill in the blank

$$1+2+4+\cdots+2^n+2^{n+1} = \underbrace{\ldots\ldots\ldots\ldots\ldots}_{\text{by inductive hypothesis}} +2^{n+1}.$$

(c) Plug $n=3$ into the predicate and simplify both sides of the equation.

(d) What are the three key steps of an inductive proof?

(e) Compare the two sides of the equation for $P(1)$, $P(2)$, $P(3)$, $P(4)$, and $P(5)$.

(Exercise on page 85.)

Hint for Exercise 74 (Asymptotic growth)**.**

By "eventually" we mean that there exists some $b \in \mathbb{N}$ such that $\forall n \in \mathbb{N}(n \geq b \implies n! > 2^n)$. To find such a b, try to compare both sides of the inequality for the first few values of n.

(Exercise on page 87.)

Hint for Exercise 75 (Asymptotic growth II)**.**

It is not immediately apparent how we can make use of the inductive hypothesis, so we have to create an opportunity to do so. Note that $(n+1)^m > n^m$.

(Exercise on page 88.)

Hint for Exercise 76 (Convergence)**.**

(a) For example, $6!! = 6 \cdot 4 \cdot 2$ and $7!! = 7 \cdot 5 \cdot 3 \cdot 1$.

(b) Try to express $(n+2)!!$ in terms of $n!!$.

(c) You can use your results from part (a) to speed up the computation.

(d) You can use your results from part (b) to transform the numerator and denominator.

(e) Use the recursion from part (d) to express a_{n+1} in terms of a_n, so that you can substitute the inductive hypothesis.

(Exercise on page 89.)

Hint for Exercise 77 (Convergence II).

(a) For example,
$$\prod_{k=1}^{3}\left(1+\frac{(-1)^n}{n+1}\right)=\left(1-\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1-\frac{1}{4}\right)=\frac{1}{2}.$$

(b) For the even terms p_{2n}, try to express the numerator and denominator in terms of n.

(c) Express p_{2n+1} in terms of p_{2n-1}, so that you'd be able to substitute the inductive hypothesis.

(d) You may use the results of the previous part!

(e) What is $\lim_{n\to\infty} p_{2n-1}$? What about $\lim_{n\to\infty} p_{2n}$?

(Exercise on page 90.)

Hint for Exercise 78 (Recurrence).

You will need two base cases.

For the inductive step, use the recursive formula $a_n = a_{n-1} + a_{n-2}$ together with the inductive hypothesis applied to each summand.

(Exercise on page 91.)

Hint for Exercise 79 (Remainder modulo 3).

You will need three base cases.

For the inductive step let's say you want to prove that the property holds for n. If you apply the inductive hypothesis to $n-3$, what do you get?

Notice that n and $n-3$ have the same remainder when divided by 3.

(Exercise on page 92.)

Hint for Exercise 80 (Making change)**.**

One way to define the predicate $P(n)$ is: there exist nonnegative integers s, t, u such that $n = 15s + 10t + 6u$. One then wants to prove $\forall n \in \mathbb{N}.[(n \geq 30) \implies P(n)]$.

You will need at least six base-cases.

(Exercise on page 93.)

Hint for Exercise 81 (Fibonacci)**.**

You will need two base cases. For the inductive step, a useful identity to note is

$$\frac{(1+\sqrt{5})^2}{2} = \frac{6+2\sqrt{5}}{2} = 3+\sqrt{5}.$$

(Exercise on page 94.)

Hint for Exercise 82 (Divisibility).

The odd numbers are $1, 3, 5, 7, \ldots$, which can be written as $2 \cdot 1 - 1, 2 \cdot 2 - 1, 2 \cdot 3 - 1, 2 \cdot 4 - 1$.

To prove the inductive step, try to find a connection between $a^{n+2} + b^{n+2}$ and $a^n + b^n$. If you'd like another hint, we show such a connection for $n = 3$ below, but make sure to try finding it yourself first!

Note that $(a^5 + b^5) = (a^3 + b^3)(a^2 + b^2) - a^2 b^2 (a + b)$.

(Exercise on page 95.)

Hint for Exercise 83 (Maximum and Minimum)**.**

Recall Definition 4.35 from the recommended reading (p. 59):

> "For $A \subseteq \mathbb{R}$, $m \in A$ is called a **maximum** (or **greatest element**) of A if for all $a \in A$, we have $a \leq m$. Similarly, $m \in A$ is called a **minimum** (or **least element**) of A if for all $a \in A$, we have $m \leq a$."

The set $\mathbb{N}$ has a minimum element but no maximum element. We can use the definition of a maximum to construct a proof by contradiction: suppose m is a maximum, and find some $n \in \mathbb{N}$ such that $n > m$.

(Exercise on page 97.)

Hint for Exercise 84 (Spot the error).

Recall the Well-Ordering Principle states that every nonempty subset of the natural numbers has a least element. Using the Well-Ordering Principle is therefore sometimes referred to as finding the minimal counterexample.

(Exercise on page 98.)

Hint for Exercise 85 (Well-Ordering from Induction)**.**

Complete induction would help here! Suppose for some $n \in \mathbb{N}$ we know that for all natural numbers $k < n$ we have $P(k)$. Can you prove $P(n)$?

Continuing from above: suppose towards contradiction that $n \in S$. Can you show that this would force n to be a least element?

(Exercise on page 99.)

Hint for Exercise 86 (Induction from Well-Ordering).

Can 1 be the minimal element of S^c?

If m is the minimal element of S^c, what can you say about $m-1$?

(Exercise on page 100.)

Hint for Exercise 87 (Using the Well-Ordering Principle)**.**

Using the well-ordering principle, one often starts with a set S of counter-examples to the statement we're trying to prove. Our goal is to show this set is empty, i.e. there are no counter-examples.

To do so, we suppose towards contradiction the set is not empty, and use the well-ordering principle to find the **minimal counter-example**. We then derive a contradiction to minimality.

Consider the set $S = \{n \in \mathbb{N} : 2 + 4 + \cdots + 2n \neq n(n+1)\}$. For the sake of contradiction, assume S is nonempty. Can you apply the well-ordering principle to S to find the least element of S? What values can the least element take? Try to consider two cases for the least element.

(Exercise on page 101.)

Hint for Exercise 88 (Division with remainder)**.**

(a) Try to show that $n \in S$.

(b) It is clear that $r \geq 0$ (why?). To prove that $r \leq m-1$, suppose towards contradiction that $r \geq m$ and show that $r - m \in S$.

(c) Summarize the proof so far.

(d) Suppose towards contradiction that $q \neq q'$, say $q > q'$. Then $(q - q')m = r - r'$, why is this a contradiction? Remember the bounds on r, r'.

(Exercise on page 102.)

Hint for Exercise 89 (Spot the error II).

If f_{s-1}, f_{s-2} were even, then it is true that $f_{s-1} + f_{s-2}$ is also even; so the error must lie somewhere with that assumption. Can you spot it?

(Exercise on page 103.)

Hint for Exercise 90 (Roundabout)**.**

Let S be the set of all possible round-trips. Prove that S is nonempty (start anywhere and keep going!) and consider its minimal element R.

(Exercise on page 104.)

Hint for Exercise 91 (Describing Relations)**.**

Recall Definition 7.1 from the recommended reading:

> "Let A and B be sets. A **relation** R **from** A **to** B is a subset of $A \times B$. If R is a relation from A to B and $(a, b) \in R$, then we say that a **is related to** b and we may write aRb in place of $(a, b) \in R$. If R is a relation from A to the same set A, then we say that R is a **relation on** A."

(Exercise on page 105.)

Hint for Exercise 92 (Properties of Relations).

Recall Definition 7.25 from the recommended reading:

> “Let R be a relation on the set A.
>
> (a) The relation R is **reflexive** if for all $a \in A$, aRa.
>
> (b) The relation R is **symmetric** if for all $a, b \in A$, if aRb, then bRa.
>
> (c) The relation R is **transitive** if for all $a, b, c \in A$, if aRb and bRc, then aRc.”

(Exercise on page 106.)

Hint for Exercise 93 (Describing Properties of Relations).

The previous exercises, Exercise 91 and Exercise 92, can be helpful. We remind the reader again of the Definition 7.25 from the recommended reading:

> "Let R be a relation on the set A.
>
> (a) The relation R is **reflexive** if for all $a \in A$, aRa.
>
> (b) The relation R is **symmetric** if for all $a, b \in A$, if aRb, then bRa.
>
> (c) The relation R is **transitive** if for all $a, b, c \in A$, if aRb and bRc, then aRc."

(Exercise on page 107.)

Hint for Exercise 94 (Counting Relations)**.**

Once again, Definition 7.1 and Definition 7.25 from the recommended reading are useful (see previous hints).

Note that *any* subset of $A \times B$ is a relation from A to B.

If R is a reflexive relation on $\{1, 2, 3\}$ then what ordered pairs must be present in A? Can there be any other ordered pairs in addition to these? Which ones? How many options are there?

(Exercise on page 108.)

Hint for Exercise 95 (Weak ordering).

In each case, we must prove that the relation is reflexive, antisymmetric, and transitive.

For example, to show that $\leq$ on $\mathbb{N}$ is antisymmetric, let $a, b \in \mathbb{N}$ be arbitrary and suppose $a \leq b$ and $b \leq a$. Then $a = b$.

(Exercise on page 109.)

Hint for Exercise 96 (Strict ordering)**.**

(a) We need to prove that $<$ is asymmetric and transitive. The proofs you wrote for Exercise 95 may be useful.

(b) Let $a \in A$ and suppose aRa. What does asymmetry imply (substitute $b = a$ in the definition)?

(c) We know that R is reflexive, antisymmetric, and transitive. We need to prove that S is asymmetric and transitive. This comes down to analyzing the cases in the definition of S.

For example, to prove that S is asymmetric suppose $a, b \in A$ are such that aSb and suppose towards contradiction that bSa. By the definition of S, from aSb we deduce aRb and $a \neq b$. From bSa we deduce bRa and $a \neq b$. What does the antisymmetry of R give?

To prove transitivity you will need to rely on asymmetry, so it's important to prove these properties in this order!

(Exercise on page 110.)

Hint for Exercise 97 (Real-world relations).

You have already determined the properties of these relations in the previous handout, you may wish to revisit that exercise (Exercise 2). Recall that an **equivalence relation** is a relation that is reflexive, symmetric, and transitive.

If $\sim$ is an equivalence relation, then any $s \in S$ determines an **equivalence class**, denoted $[s]$, which is the set of all elements related to s, that is $[s] := \{x \in S : x \sim s\}$.

(Exercise on page 111.)

Hint for Exercise 98 (String length).

You need to prove that the relation is reflexive, symmetric and transitive.

To describe equivalence classes, you can start with one word, for example `cat` and think what other words would be related to it. How can you generalize your observation?

(Exercise on page 112.)

Hint for Exercise 99 (Digraphs)**.**

In each case, we need to check reflexivity, symmetry and transitivity. You may wish to revisit the previous handout on relations, where you analyzed how to use the digraph to determine each of these properties.

Recall that reflexivity can be verified by checking that each node has a self-loop.

Symmetry can be verified that each connected pair of nodes have a "two-way" connection, aRb and bRa.

For transitivity, one needs to check that for if one element is connected to two other elements, then those two elements are connected to each other as well.

(Exercise on page 113.)

Hint for Exercise 100 (Common Misconception).

It is true that from $a \sim b$ and $b \sim a$ we can conclude $a \sim a$.

It is true that if $a \sim b$ then $b \sim a$.

(Exercise on page 114.)

Hint for Exercise 101 (Multifunctional)**.**

It may help to revisit the previous exercise, Exercise 100.

Even if you do not remember your Linear Algebra, this is still a worthwhile exercise! You can think of $n = 1$, which just corresponds to real numbers instead of matrices.

Start with reflexivity and symmetry, then multifunctionality. What does this tells us about transitivity?

(Exercise on page 115.)

Hint for Exercise 102 (Remainders).

(a) Start by writing $a = mq + r$ and $b = mq' + r'$. Suppose $r = r'$ and prove that $m|(b-a)$.

Conversely, suppose $m|(b-a)$ so that $b - a = q''m$. We have $q''m = b - a = m(q' - q) + (r' - r)$. Rearrange so that $m(q'' - q' + q) = r' - r$ and take absolute values so that $m|q'' - q' + q| = |r' - r|$. Argue that the right side is strictly less than m, so that both sides are 0.

(b) Check reflexivity, symmetry, and transitivity. It will be easier to use the definition of M (rather than part (a)), but be careful to note where we use the uniqueness of the remainder.

(c) Two numbers are related if and only if they share a remainder.

(d) If a and b end in the same digit, then $b - a$ is divisible by 10.

(Exercise on page 116.)

Hint for Exercise 103 (Advanced Mathematics).

(a) We need to prove that R is reflexive, transitive, and symmetric.

Reflexivity follows from the fact that $0 \in \mathbb{Z}$.

Symmetry follows from the fact that if $z \in \mathbb{Z}$, then so is its inverse $-z$.

Transitivity follows from the fact that if $z, z' \in \mathbb{Z}$, then so is their sum $z + z'$.

None of the real numbers in $[0, 1)$ is related to any of the others.

The real numbers in $[0, 1)$ determine a complete system of representatives of the equivalence classes.

(b) The proof that Q is an equivalence relation is analogous to the proof that R is an equivalence relation.

The equivalence classes are hard to imagine! There is one equivalence class for all the rational numbers $\mathbb{Q}$.

In general, if r is irrational, it defines an equivalence class $[r]$ all of whose elements are also irrational.

(Exercise on page 117.)

Hint for Exercise 104 (Counting).

(a) A partition of A is a collection Ω of subsets of A satisfying three conditions.

Each subset must be nonempty, the subsets are pairwise disjoint, and the subsets cover the set A.

(b) A partition of A is a collection of *nonempty* subsets of A.

(c) There is only one nonempty subset of $\{1\}$. On the other hand, $\{1, 2\}$ has 3 nonempty subsets.

Of the three nonempty subsets of $\{1, 2\}$ we need to choose a collection of them that is pairwise disjoint and covers $\{1, 2\}$. There are precisely 2 ways to choose such a collection.

(d) There are 5 different partitions of $\{1, 2, 3\}$.

(Exercise on page 119.)

Hint for Exercise 105 (Find the Partitions)**.**

Remember the analogy between a partition and a pie chart. There are three conditions that the slices should satisfy. You can draw a diagram where each block of the partition (each set of Ω) is a slice of the pie chart.

A **partition** of a set A is a collection Ω of *nonempty* subsets of A that are *pairwise disjoint* and *cover* A.

Exactly half of $\Omega_1, \ldots, \Omega_6$ are partitions of A_1.

Exactly half of $\Omega_7, \ldots, \Omega_{10}$ are partitions of A_2.

(Exercise on page 120.)

Hint for Exercise 106 (Find the Partitions II).
Recall that the Cartesian product of two sets A and B is defined as $A \times B = \{(a, b) | a \in A, b \in B\}$

Exactly one of $\Omega_1, \ldots, \Omega_5$ is a partition of $\mathbb{Z} \times \mathbb{Z}$.

Recall that intervals in $\mathbb{R}$ are defined by

$$\begin{aligned}(a, b) &= \{x \in \mathbb{R} | a < x < b\},\\ [a, b] &= \{x \in \mathbb{R} | a \leq x \leq b\},\\ (a, b] &= \{x \in \mathbb{R} | a < x \leq b\}.\end{aligned}$$

There are exactly three partitions of $\mathbb{R}$ among $\Omega_6, \ldots, \Omega_{10}$.

(Exercise on page 121.)

Hint for Exercise 107 (Constructing Partitions)**.**

Recall that a partition of A is a collection Ω of subsets of A. Each set in Ω is called a **block**. In our pie-chart analogy (cf. Exercise 104), the blocks are the "slices of the pie".

An example of a partition with two finite blocks and one infinite block is

$$\Omega = \{\{5\}, \{7\}, \mathbb{N} \setminus \{5, 7\}\}.$$

Try to come up with more examples.

An example of a partition with infinitely-many blocks is

$$\Omega = \{\{1, 2, \ldots, 10\}, \{11, 12, \ldots, 20\}, \ldots, \{91, 92, \ldots, 100\}, \{101, 102, \ldots, \{110\}\}, \ldots\}.$$

Try to come up with more examples.

An example of a partition with infinitely many blocks is

$$\Omega = \{\{1, 4\}, \{2, 5\}, \{3, 6\}, \{7, 10\}, \{8, 11\}, \{9, 12\}, \ldots\}.$$

Try to come up with more examples.

A partition of $\mathbb{N}$ is a collection of subsets of $\mathbb{N}$ that are nonempty, pairwise disjoint, and cover $\mathbb{N}$.

(Exercise on page 122.)

Hint for Exercise 108 (Relations from subsets).

Recall that a relation R on a set A is said to be an **equivalence relation** if it is reflexive, symmetric, and transitive. It may help to revisit the previous handout where you practiced determining whether a relation given as a digraph is an equivalence relation.

Note that, for example, $0R_{\Omega_1}0$ because there is some $X \in \Omega_1$ such that $0 \in X$. Similarly, as long as a is an element of some $X \in \Omega$ we always have $aR_{\Omega}a$, so that reflexivity can only fail if Ω does not cover A.

Note that if $aR_{\Omega}b$ then there is some $X \in \Omega$ such that $a, b \in X$. Then for that very same X we have $b, a \in X$, so $bR_{\Omega}a$. Thus, symmetry can never fail.

Only two out of $R_{\Omega_1}, \ldots, R_{\Omega_6}$ fail to be an equivalence relation.

(Exercise on page 123.)

Hint for Exercise 109 (Relations and Partitions)**.**

(a) Suppose $aR_\Omega b$, so that there is some $X \in \Omega$ such that $a, b \in X$. What does it mean to prove $bR_\Omega a$?

(b) We have $aR_\Omega a$ if and only if there is some $X \in \Omega$ such that $a \in X$. Recall that Ω covers A if and only if $\bigcup_{X\in\Omega} X = A$. What is the definition of the "big union"?

(c) Suppose the sets in Ω are pairwise disjoint and that $aR_\Omega b$ and $bR_\Omega c$. Then there is some $X \in \Omega$ such that $a, b \in X$ and some $Y \in \Omega$ such that $b, c \in Y$. What is the connection between X and Y?

(d) One such example is $A = \{1, 2, \ldots, 10\}$ and $\Omega = \{\{1\}, A\}$. Can you think of other examples?

(e) If Ω is a partition then its sets are nonempty, cover A, and are pairwise disjoint. On the other hand R_Ω is an equivalence relation if and only if it is reflexive, symmetric, and transitive.

(f) The equivalence classes are precisely the blocks in the partition. You can prove this by showing that if Ω is a partition then for every $a \in A$ there must be exactly one set X_a in Ω which has a as an element. Then show that $X_a = [a]$.

(Exercise on page 124.)

Hint for Exercise 110 (Relations and Partitions II)**.**

We need to prove that the equivalence classes are nonempty, cover A, and are pairwise disjoint. The first two conditions follow from the fact that R is reflexive.

To prove the last condition, you need to show that if $c \in [a] \cap [b]$ then $[a] = [b]$. This follows from the symmetry and transitivity of R.

Suppose $x \in [a]$ prove that $x \in [b]$. We have xRa, aRc, cRb.

(Exercise on page 125.)

Hint for Exercise 111 (Refinements).

(a) Start by checking that Ω_1, Ω_2 are indeed partitions, for otherwise the question does not make sense.

Next, check whether each block in Ω_1 is a subset of a block in Ω_2.

For example, $\{1, 2\} \subseteq \{1, 2, 3\}$.

(b) There are many correct solutions!

For example, $\{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\}$ is always a refinement of any partitions of $A = \{1, 2, 3, 4, 5, 6\}$, do you see why? Try to come up with a different example and explain the general procedure of generating examples.

In general, we can generate examples by taking one or more blocks of Ω_2 and partitioning it.

(c) Suppose $X \in P_1$ is arbitrary. Can we fine $Z \in P_3$ such that $X \subseteq Z$?

Use P_2 as an intermediary step. This follows the familiar patterns of proofs of transitivity.

(d) The blocks of Q_1 are "smaller" than the blocks of Q_2, so *less* elements are related to each other.

Prove that $R_{Q_1} \subseteq R_{Q_2}$.

This is equivalent to showing that for any $x, y \in A$ we have $xR_{Q_1}y$ implies $xR_{Q_2}y$.

(Exercise on page 126.)

Hint for Exercise 112 (Representatives)**.**
To prove that the three statements are equivalent, suffice it to prove $(a) \implies (b) \implies (c) \implies (a)$.

Recall the definition of equivalence class:

$$[x] := \{y \in A \,:\, yRx\}\,.$$

To prove that $(a) \implies (b)$, suppose $[a] = [b]$ and prove that $a \in [b]$. This follows from the reflexivity of R.

To prove that $(b) \implies (c)$, suppose $a \in [b]$ and prove that aRb. This is the definition of the equivalence class $[b]$.

To prove that $(c) \implies (a)$, suppose aRb. We need to prove that $[a] \subseteq [b]$ and $[b] \subseteq [a]$. Start with the former.

To prove $[a] \subseteq [b]$, let $x \in [a]$ be arbitrary. Use the transitivity of R to conclude $x \in [b]$.

To prove $[b] \subseteq [a]$, note that by the symmetry of R we have aRb implies bRa, so by the very same proof that $[a] \subseteq [b]$ we can now conclude $[b] \subseteq [a]$.

(Exercise on page 127.)

Hint for Exercise 113 (Operations)**.**
Throughout, suppose $[a] = [a']$ and $[b] = [b']$.

(a) The operation $\oplus_1$ is well-defined. We need to check that

$$[a] = [a] \oplus_1 [b] = [a'] \oplus_1 [b'] = [a']$$

which is clear. Explain the leftmost and rightmost equalities in detail.

(b) The operation $\oplus_2$ is well-defined. We need to check that $a + b$ and $a' + b'$ have the same parity; that is, that $2|(a' + b' - (a + b))$. Use the definition of $\equiv_{10}$.

(c) The operation $\oplus_3$ is *not* well-defined. One counter-example is $[0] = [10]$ and $[1] = [11]$. Try to find more counter-examples and explain why they are counter-examples.

(d) The operation $\oplus_4$ is *not* well-defined. One counter-example is $[0] = [10]$ and $[1] = [1]$. Try to find more counter-examples and explain why they are counter-examples.

(e) The operation $\oplus_5$ is well-defined. We need to prove that $[2a + 3b] = [2a' + 3b']$, which is the same as proving $10|(2a' + 3b' - (2a + 3b)) = 2(a' - a) + 3(b' - b)$. Use the definition of $\equiv_{10}$.

(Exercise on page 128.)

Hint for Exercise 114 (Properties)**.**
Let $A, B, C \in X/R$ be arbitrary equivalence classes with representatives $a, b, c \in X$.

In each case the proof the is the same: pass from $\oplus$ to $\boxplus$, and use the property of $\boxplus$.

For example, to prove commutativity

$$\begin{aligned} A \oplus B &= [a] \oplus [b] \\ &= [a \boxplus b] \\ &= [b \boxplus a] \\ &= [b] \oplus [a] \\ &= B \oplus A. \end{aligned}$$

Be sure to explain each equality in the chain.

If o is the identity element of $\boxplus$, then $[o]$ is the identity element of $\oplus$.

If a' is the $\boxplus$-inverse of a, then $[a']$ is the $\oplus$-inverse of $[a]$.

(Exercise on page 129.)

Hint for Exercise 115 (Integers)**.**

(a) Show that $\sim$ is reflexive, symmetric, and transitive.

For transitivity, let $(a, b), (c, d), (e, f) \in Z$ and suppose $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$. Translate these to equations and then prove that $a + f = b + e$.

(b) We have $(a, b) \in [(a - b, 0)]$ if $a \geq b$, and $(a, b) \in [(0, b - a)]$ otherwise.

Prove that $[(a, 0)] \neq [(0, b)]$ and that if $x \neq y$ then $[(x, 0)] \neq [(y, 0)]$ and $[(0, x)] \neq [(0, y)]$.

(c) Suppose $[(a, b)] = [(a', b')]$ and $[(c, d)] = [(c', d')]$. Prove that

$$[(a, b)] \boxplus [(c, d)] = [(a', b')] \boxplus [(c', d')].$$

(d) The numbers $n, m \in \mathbb{N} \cup \{0\}$ are identified with $[(n, 0)]$ and $[(m, 0)]$ and their sum $n + m$ with $[(n + m, 0)]$. Show that

$$[(n, 0)] \boxplus [(m, 0)] = [(n + m, 0)].$$

(e) Let $n \in \mathbb{N} \cup \{0\}$. Then n is identified with $[(n, 0)]$ and $-n$ with $[(0, n)]$. Show that

$$[(n, 0)] \boxplus [(0, n)] = [(0, 0)].$$

(f) Use the technique of the previous part to show that $5 + (-2) = 3$ and that $2 + (-5) = -3$.

(g) Recall that $[(a, b)]$ is identified with n if and only if $a \geq b$ and $n = a - b$ (in which case $[(a, b)] = [(a - b, 0)] = [(n, 0)]$).

(h) Suppose $[(a,b)] = [(a',b')]$. Prove that $\boxminus[(a,b)] = \boxminus[(a',b')]$. You may wish to use Exercise 112 to show that the representatives are related by $\sim$.

(i) Show that $e = a + d$ and $f = b + c$.

The binary $\boxminus$ is well-defined because the binary $\boxplus$ and unary $\boxminus$ are well-defined. Suppose $[(a,b)] = [(a',b')]$ and $[(c,d)] = [(c',d')]$. Use the fact that $\boxplus$ and $\boxminus$ are well-defined to argue that we must have

$$[(a,b)] \boxminus [(c,d)] = [(a',b')] \boxminus [(c',d')].$$

You should get $[(2,0)] \boxminus [(5,0)] = [(2,5)]$. Argue that $(2,5)$ is a representative of the equivalence class of -3.

(Exercise on page 131.)

Hint for Exercise 116 (Rationals)**.**

(a) Prove that $\sim$ is reflexive, symmetric, and transitive.

For transitivity, suppose $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$. This gives rise to the equation $adcf = bcde$. It is possible to cancel d since $d \neq 0$, but for c one has to consider the two cases that $c \neq 0$ and that $c = 0$.

(b) Suppose $[(a, b)] = [(a', b')]$ and $[(c, d)] = [(c', d')]$. Prove that $[(ac, bd)] = [(a'c', b'd')]$. By the definition of $\sim$, this is equivalence to showing that $acb'd' = bda'c'$.

(c) The integers $z, z' \in \mathbb{Z}$ are identified with the equivalence classes $[(z, 1)], [(z', 1)]$ (respectively); their product zz' is identified with $[(zz', 1)]$. Prove that

$$[(z, 1)] \otimes [(z', 1)] = [(zz', 1)].$$

(d) The integer z is identified with $[(z, 1)]$ and $\frac{1}{z}$ with $[(1, z)]$. Prove that $[(z, 1)] \otimes [(1, z)] = [(1, 1)]$.

(e) For example, $[(1, 1)] = [(2, 2)]$.

(f) Suppose $[(a, b)] = [(a', b')]$ and $[(c, d)] = [(c', d')]$. Prove that $(ad + bc, bd) \sim (a'd' + b'c', b'd')$.

(g) Show that $[(z, 1)] \oplus [(z', 1)] = [(z + z', 1)]$.

(h) Suppose $[(a, b)] = [(a', b')]$ and show that $(b, a) \sim (b', a')$.

(i) The binary $\div$ is well-defined because $\otimes$ and the unary $\div$ are well-defined. Suppose $[(a,b)] = [(a',b')]$ and $[(c,d)] = [(c',d')]$. Use the fact that $\div$ is well-defined to conclude that $\div[(c,d)] = \div[(c',d')]$. Now use the fact that $\otimes$ is well-defined.

(j) Show that $[(a,1)] \div [(b,1)] = [(a,b)]$.

More generally, show that $[(r,s)] \div [(t,u)] = [(ru,st)]$.

(Exercise on page 132.)

Hint for Exercise 117 (Non Functions)**.**

Recall Definition 8.1 from the Recommended Reading:

> "Let X and Y be two nonempty sets. A **function** f **from** X **to** Y is a relation from X to Y such that for every $x \in X$, there exists a unique $y \in Y$ such that $(x, y) \in f$."

The definition has two parts "there exists" and "unique" and each of these parts may fail.

There may be an element in the domain which does not get matched to any element *in the codomain*; this is the way in which "there exists" fails.

There may be an element in the domain which gets matched to more than one element in the codomain; this is the way in which "unique" fails.

(a) Look for failure of "there exists".

(b) Look for failure of "unique".

(c) Look for failure of "there exists".

(d) Look for failure of "there exists".

(e) Look for failure of "there exists".

(f) Look for failure of "unique".

(Exercise on page 133.)

Hint for Exercise 118 (Function Construction).

(a) Remember that a relation from A to B is a subset of $A \times B$; how many subsets are there?

(b) Remember, to build a function, for each element in the domain, you must choose exactly one element from the codomain. How many possibilities are there?

(c) There are two ways a relation can fail to be a function. What are they? Try to construct a function that fails one condition but satisfies the other and another that does the opposite.

(d) For each element of A, you can choose any element of B. Count how many options you have for the pairs in your function.

(Exercise on page 134.)

Hint for Exercise 119 (Is This a Function?).

In each case, we must check that each element in the domain is related to exactly one element in the codomain.

(a) Verify that this is a function.

(b) One element is unmatched, which one?

(c) Verify that this is a function.

(d) Some elements are matched more than once, can you find such an example?

(e) Verify that this is a function.

(f) One element is matched more than once, which one?

(g) Verify that this is a function.

(h) Verify that this is a function.

(Exercise on page 135.)

Hint for Exercise 120 (Domain and Range).

Recall Definition 8.1 from the Recommended Reading:

> "Let X and Y be two nonempty sets. A **function** f **from** X **to** Y is a relation from X to Y such that for every $x \in X$, there exists a unique $y \in Y$ such that $(x, y) \in f$.
>
> The set X is called the **domain** of f and is denoted by $\mathrm{Dom}(f)$. The set Y is called the **codomain** of f and is denoted by $\mathrm{Codom}(f)$ while the subset of the codomain defined via
>
> $$\mathrm{Rng}(f) := \{y \in Y \mid \text{ there exists } x \text{ such that } (x, y) \in f\}$$
>
> is called the **range** of f or the **image** of X under f."

To determine the domain of a function, figure out what inputs are valid for the function. Similarly, to determine the range of a function, figure out the set of actual outputs (for all the possible inputs).

(a) Any nonnegative integer is a possible input, what are the outputs?

(b) We have $f(a) = 1$ and $f(z) = 26$.

(c) For example, $\max((1, 2)) = \max((2, 2)) = 2$.

(d) For example, $f(111101) = 1$, $f(101010) = f(000) = 3$, and $f(111) = 0$.

(e) Note that $(\sqrt{r})^2 = r$. When does this equality make sense?

(Exercise on page 136.)

Hint for Exercise 121 (Codomain versus Range)**.**
Revisit Definition 8.1 from the recommended reading.

(a) What values can $x^2 + 1$ actually take when x runs over all the numbers in the domain of f?

Note that $x^2 + 1 \geq 1$. Conversely, if $y \geq 1$, find a value x such that $x^2 = y$.

Can you find elements in the codomain that are not in the range of f?

(b) Note that g has the same rule as f, the only difference is in the specified codomain.

(c) Think about what makes f and g different functions.

(Exercise on page 137.)

Hint for Exercise 122 (Special Functions).

(a) For the inclusion map $\iota : A \to B$, does every input in A have exactly one image in B? What if we reverse the direction, would every element of B have an image in A?

(b) Compare the two maps carefully. Remember that maps with different domains and/or codomains are different maps.

(c) Recall the domain is the set of allowed inputs to the function, the codomain is where the possible outputs live, and the range is the set of actual outputs for the function.

(d) Note that the range of a function must always be a subset of its codomain.

(Exercise on page 138.)

Hint for Exercise 123 (Piecewise-Defined Functions)**.**

(a) Note that $x = 1$ is both ≤ 1 and also ≥ 1. On the other hand, 0 is neither > 0 nor < 0.

(b) There are infinitely many ways to revise the conditions in the definition of f to make the function well-defined. For example, one can make all the conditions strict and define $f(1) = 1$ and $f(0) = 0$. Try to make as few changes as possible, so you don't have to define $f(0)$ and $f(1)$ as new conditions.

Argue that your modified function is indeed a function by showing that every element in the domain is matched with at least one value and at most one value in the codomain.

(c) What happens if $a \in A$ is not an element of A_1, A_2, A_3? What condition must A_1, A_2, A_3 satisfy in order to ensure this cannot happen?

What happens if $a \in A$ is an element of both A_1 and A_2? What conditions do we need to resolve this ambiguity? Hint: we don't *have* to demand that A_1 and A_2 are disjoint.

(Exercise on page 139.)

Hint for Exercise 124 (The Ceiling and Floor Functions).

Make sure you understand the definitions of the ceiling and floor functions. For example, $\lfloor 5.5 \rfloor = 5$ while $\lceil 5.5 \rceil = 6$. What about negative numbers?

On the other hand, $\lfloor -5.5 \rfloor = -6$ while $\lceil -5.5 \rceil = -5$.

What happens at each integer? What about $\lfloor x \rfloor$ for $x \in [0, 1)$? If you need to, you can try evaluating more examples by hand before trying to sketch the graphs.

Note that $\lfloor x \rfloor$ and $\lceil x \rceil$ are (different) constant for $x \in [n, n+1)$ where $n \in \mathbb{Z}$.

To prove $r - \lfloor r \rfloor \in [0, 1)$, start by reasoning that $\lfloor r \rfloor \leq r$. Suppose for contradiction that $r - \lfloor r \rfloor \geq 1$ and show that $\lfloor r \rfloor$ is not the greatest integer less than or equal to r.

To prove existence, use the previous part. To prove uniqueness, suppose that $r = n + \theta = n' + \theta'$ and rearrange to $n - n' = \theta' - \theta$. Note that $n - n' \in \mathbb{Z}$; to what interval does $\theta' - \theta$ belong?

(Exercise on page 140.)

Hint for Exercise 125 (Functions and Equivalence Relations).

In each case one must check that if $[x] = [y]$ then $f([x]) = f([y])$. Recall that $[x] = [y]$ if and only if $x \sim y$.

(a) The function is not well-defined. Give an example where $[x] = [y]$ but $f([x]) \neq f([y])$.

(b) The function is well-defined, suppose $[x] = [y]$ and prove that $f([x]) = [x] = [y] = f([y])$.

(c) The function is well-defined. Suppose $[x] = [y]$, so that $10|(y - x)$. Show that $f([x]) = f([y])$.

(d) The function is well-defined. If $[(x, y)] = [(x', y')]$, prove that $f[(x, y)] = f[(x', y')]$.

(e) The function is not well-defined. Give an example where $[x] = [y]$ but $f([x]) \neq f([y])$.

(f) The function is well-defined. Use Exercise 124(f) to show that if $[x] = [y]$ then $x - \lfloor x \rfloor = y - \lfloor y \rfloor$.

Suppose $x = \lfloor x+ \rfloor + \theta_x$ and $y = \lfloor y+ \rfloor + \theta_y$ with $\theta_x, \theta_y \in [0, 1)$. Conclude that

$$\theta_y - \theta_x = (y - x) - (\lfloor y \rfloor - \lfloor x \rfloor).$$

Examine the proof of 124(f).

(Exercise on page 141.)

Hint for Exercise 126 (Some Properties of the Ceiling and Floor Functions).

(a) Write $x = n + \theta$ according to Exercise 124(f), and distinguish between the cases that $\theta < 1/2$ and $\theta \geq 1/2$.

For example, if $\theta \geq 1/2$, then $2x = (2n + 1) + (2\theta - 1)$ is the unique expression of $2x$ as the sum of an integer and a real in the unit interval (why?). In particular, $\lfloor 2x \rfloor = 2n + 1$ (why?).

(b) The statement is false. Try some familiar fractions to find counterexamples.

(c) The statement is false, for similar reasons to the previous part.

(d) Write $x = n + \theta$ according to Exercise 124(f) and distinguish between the cases that n is even or odd.

For example, if $n = 2k + 1$, then $\frac{x}{2} = k + (\frac{1}{2} + \frac{\theta}{2})$ is the unique expression of $\frac{x}{2}$ as the sum of an integer and a real number in the (half-open) unit interval. It follows that $\lfloor x/2 \rfloor = k$ and $\lceil x/2 \rceil = k + 1$ (why?).

(Exercise on page 142.)

Hint for Exercise 127 (Basic definitions).

Start with Definition 8.26 from the recommended reading:

> "Let $f : X \to Y$ be a function.
>
> (a) The function f is said to be **injective** (or **one-to-one**) if for all $y \in \mathrm{Rng}(f)$, there is a unique $x \in X$ such that $y = f(x)$.
>
> (b) The function f is said to be **surjective** (or **onto**) if for all $y \in Y$, there exists $x \in X$ such that $y = f(x)$.
>
> (c) If f is both injective and surjective, we say that f is **bijective**."

The definition of a function requires that for every element in the domain there is a unique element in the codomain that is matched to it. The definition says nothing about the elements in the codomain.

(a) If every element in the codomain has at most one element in the domain matching to it, the function is said to be injective.

(b) If every element in the codomain has at least one element in the domain matching to it, the function is said to be surjective.

(c) If every element in the codomain has exactly one element in the domain matching to it, the function is said to be bijective.

(a) Neither.

(b) Surjective.

(c) Neither.

(d) Neither.

(e) Bijective.

(f) Injective.

(g) Surjective.

(h) Injective.

(i) Injective.

(Exercise on page 143.)

Hint for Exercise 128 (Classifying functions).

Revisit the definitions from the previous exercise.

The "horizontal line test" can also be useful for some of these functions.

To prove injectivity, suppose $f(x) = f(x')$ and prove $x = x'$.

To prove surjectivity, start with an arbitrary element y in the codomain and find an element x in the domain such that $f(x) = y$.

(Exercise on page 144.)

Hint for Exercise 129 (Piecewise-defined Function).

(a) Use the fact that $A \cap C = \emptyset$ to prove that for every $x \in A \cup C$ there is exactly one $y \in B \cup D$ with $h(x) = y$. It may help to review Exercise 7 from the Introduction to Functions handout.

(b) Even if f, g are injective, it does not necessarily follow that h is injective.

(c) If f, g are surjective then so is h. Given an arbitrary $y \in B \cup D$, use the surjectivity of f, g to find a suitable $x \in A \cup C$ such that $h(x) = y$.

(d) Yes, h would be bijective! You already know that h must be surjective, so it remains to show that it must also be injective.

Given $x, x' \in A \cup C$ such that $h(x) = h(x')$, prove that both x, x' are elements of A or both are elements of C, then use the injectivity of f, g.

(Exercise on page 145.)

Hint for Exercise 130 (Finite sets)**.**

(a) There are several ways of constructing such a function, but perhaps the easiest is to argue that the inclusion map works in this case.

(b) You can almost use the identity map, you need to assign the "extra" elements to some arbitrary $b \in B$; what element is guaranteed to be in B regardless of the value of m?

(c) See part (a); what is the inclusion map called in this case?

(d) Argue that $B \subseteq \bigcup_{a \in A} \{f(a)\}$ and therefore m is at most $\sum_{a \in A} |\{f(a)\}|$.

(e) This part is a bit challenging, but we can take a similar approach to the previous part. For each $b \in B$, let S_b be the set $\{a\}$ if there is some $a \in A$ such that $f(a) = b$ (argue there can be at most one such a) and $S_b = \emptyset$ otherwise. Argue that $A = \bigcup_{b \in B} S_b$ and therefore n is at most $\sum_{b \in B} |S_b|$.

(f) Combine the two previous parts.

(g) All of these assertions follow from the various parts of the question. For example, the first assertion follows from parts (a) and (e) taken together.

(Exercise on page 146.)

Hint for Exercise 131 (Constructing a bijection)**.**

(a) The first element of the first row is $(1,1)$ and its count 1; the first element of the second row is $(2,1)$ and its count is $n+1$ (because we have counted n elements before it); what is the count of the first element of the third row $(3,1)$? Can you find a formula for $(r,1)$?

(b) The element (r,k) is the k-th element of row r. You already have a formula for the first element $(r,1)$, how can we reach the k-th element?

(c) Use your formula from part (b) above.

(d) To show that Φ is injective, suppose $\Phi((r,k)) = \Phi((r',k'))$, so that $(r-1)n+k = (r'-1)n+k'$. Rearranging and taking absolute values,

$$|r-r'|n = |k'-k|.$$

Now $k, k' \in B$ so what does this tell us about the possible values of $|k'-k|$?

(e) To show that Φ is surjective, start with $1 \leq y \leq mn$ and apply Division with Remainder to write $y = rn+q$. We'd like to argue that $\Phi((r+1,q)) = y$, but this is not quite right because $(r+1,q)$ may not be an element of $A \times B$. Consider two possible cases for the value of q (and use them to also argue about the value of r).

(Exercise on page 147.)

Hint for Exercise 132 (Set difference)**.**

(a) Start with some $y \in f[A] \setminus f[B]$. This means $y \in f[A]$ and $y \notin f[B]$. Therefore, there is some $a \in A$ for which $f(a) = f[A]$. What is the corresponding statement for B?

Conclude that $a \notin B$ and therefore $a \in A \setminus B$.

(b) The next part of the question provides a strong hint for this one!

(c) For the "if" direction, suppose $x \neq x'$ and set $A = \{x\}, B = \{x'\}$. Conclude that $f(x) \neq f(x')$.

For the "only if" direction, prove that $f[A \setminus B] \subseteq f[A] \setminus f[B]$. Starting with $y \in f[A \setminus B]$ and an $a \in A \setminus B$ such that $f(a) = y$, prove that $y \notin f[B]$ by showing that for every $b \in B$ we must have $f(b) \neq y$.

(Exercise on page 148.)

Hint for Exercise 133 (Cantor's Theorem)**.**

Revisit Russell's Paradox, §3.2 of the recommended text. The argument here is quite similar!

Assume for contradiction that $x \in X$ is such that $f(x) = Y$. Determine whether x itself is an element of Y.

(Exercise on page 149.)

Hint for Exercise 134 (Composition of functions).

If f, g are functions, the composition $f \circ g$ is defined if and only if the codomain of g is a subset of the domain of f.

If the composition $f \circ g$ is defined, we the rule is given by $(f \circ g)(x) = f(g(x))$, which sometimes can be simplified!

(a) Both $f \circ g$ and $g \circ f$ are defined in this case.

(b) Once again, both $f \circ g$ and $g \circ f$ are defined; they even turn out to be the same function in this case!

(c) Neither $f \circ g$ nor $g \circ f$ is defined.

(d) Exactly one of $f \circ g$ and $g \circ f$ is defined.

(e) Careful! Neither $f \circ g$ nor $g \circ f$ is defined, but not for the "obvious reasons".

(Exercise on page 151.)

Hint for Exercise 135 (Order of composition)**.**

(a) Always try the simplest possible examples first; it will either work or could give you a hint how to proceed.

The "simplest" possible examples in this case include constant functions, the identity function, linear functions, and polynomials.

(b) Compute the compositions $f \circ g$ and $g \circ f$ and equate the results.

(c) Only the identity function $h(x) = x$ commutes with all affine functions!

Suppose $h(x)$ is not the identity. Try to compose it with a well-chosen constant function.

(Exercise on page 152.)

Hint for Exercise 136 (Compositions and injectivity).

(a) Suppose $x_1, x_2 \in X$ are such that $(g \circ f)(x_1) = (g \circ f)(x_2)$. Show that $x_1 = x_2$. Start by unpacking the definition of composition and then use the information that f, g are injective.

(b) There are infinitely many examples. Try to construct one with finite sets for simplicity. Another approach is to use familiar functions such as the squaring function x^2 or the absolute value function $|x|$ whose injectivity depends on their domain.

(c) Suppose $g \circ f$ is injective, your example from part (b) shows that g may fail to be injective, what about f?

Try to prove that f is injective. If the proof works, you are done! Otherwise, where the proof fails you will find a counterexample!

Suppose $f(x_1) = f(x_2)$. Apply g and prove that $x_1 = x_2$.

(d) Note that the identity function is always bijective! Apply part (c).

(Exercise on page 153.)

Hint for Exercise 137 (Compositions and surjectivity)**.**

This exercise is "dual" to Exercise 136, so you should be able to solve it by carefully thinking through how you solved Exercise 136.

(a) Start with $z \in Z$; apply the fact that g is surjective to obtain y; apply the fact that f is surjective to obtain x. Prove that $g \circ f$ maps x to z.

(b) There are infinitely many examples. Try to construct one with finite sets for simplicity. Another approach is to use familiar functions such as constants and inclusions whose surjectivity depends on their domain.

(c) Suppose $g \circ f$ is surjective, your example from part (b) shows that f may fail to be surjective, what about g?

Try to prove that g is surjective. If the proof works, you are done! Otherwise, where the proof fails you will find a counterexample!

Start with $z \in Z$ and find some x such that $(g \circ f)(x) = z$. Can you use this fact to find some y such that $g(y) = z$? What is the definition of composition?

(d) Note that the identity function is always bijective! Apply part (c).

(Exercise on page 154.)

Hint for Exercise 138 (Left- and right-inverses).

(a) Consult Exercise 136 and Exercise 137; recall that injectivity is a necessary condition for a left-inverse while surjectivity a necessary condition for a right-inverse.

(b) To find a left-inverse for f_1 we must find some $g_1 : \{a, b, c\} \to \{1, 2\}$ such that $g_1 \circ f_1$ is the identity function on $\{1, 2\}$; that is $1 = g_1(f_1(1)) = g_1(a)$ and so on.

The value $g_1(c)$ can be chosen arbitrarily to be either 1 or 2.

f_2 has a right-inverse; f_3 has both a right- and a left-inverse and they are the same function!

(c) If the function is injective but not surjective, we must modify the codomain.

If the function is surjective but not injective, we must modify the domain.

It is possible that we should modify both the domain and the codomain!

(Exercise on page 155.)

Hint for Exercise 139 (Inverse relation)**.**

(a) First show that $(R^{-1})^{-1} \subseteq A \times B$. Then show that $(a, b) \in R \iff (a, b) \in (R^{-1})^{-1}$ (apply the definition of the inverse relation!).

(b) Fix some $b \in B$ such that $\forall a \in A.(b, a) \notin R^{-1}$; can you translate this to a statement about R instead? What does it tell you about the function R? Can it be injective, surjective, bijective?

(c) In the previous part you've found that if it is *not* true that every $b \in B$ has at least one $a \in A$ for which $(b, a) \in R^{-1}$, then R cannot be surjective. Take the contrapositive.

(d) Suppose $(b, a); (b, a') \in R^{-1}$ show that $R(a) = R(a')$. What does this say about the function R?

(e) In the previous part you've found that if it is *not* true that every b has at most one $a \in A$ for which $(b, a) \in R^{-1}$ then R cannot be injective. Take the contrapositive.

(f) Recall that a relation is a function if and only if every element in the domain has *exactly one* element in the codomain to which it is matched.

Exactly one means "at least one" and "at most one".

(Exercise on page 156.)

Hint for Exercise 140 (Two-sided inverse)**.**

(a) If $h : \{a, b, c\} \to \{1, 2\}$ is a left-inverse for f_1, then we must have $1 = h(f_1(1)) = h(a)$ and $2 = h(f_1(2)) = h(b)$ (why?). What about c?

(b) If $h : \{a, b\} \to \{1, 2, 3\}$ is a right-inverse for f_2, then we must have $h(b) = 3$ (why?). What about $h(a)$?

(c) Note that g, h have the same domain and codomain, so it remains to prove they have the same rule; i.e. that they agree on every input.

We must find a way to use both hypotheses: the existence of a left-inverse and the existence of a right-inverse.

Recall that function composition is associative (what does this mean?). Compute $(g \circ f \circ h)(y)$ in two different ways.

(d) This follows with a little bit of thought from the previous part.

Fix one particular left-inverse. Prove that all the right-inverses are equal to your fixed left-inverse. This proves that the right-inverse is unique (why?). Now do the same for the left-inverse.

Note that any two-sided inverse is also a one-sided inverse.

(e) You've just shown that whenever a two-sided inverse exists it is unique.

Therefore, suffice it to show that $f^{-1} \circ g^{-1}$ is a two-sided inverse for $g \circ f : X \to Z$.

(Exercise on page 157.)

Hint for Exercise 141 (Cantor–Schröder–Bernstein Theorem)**.**

(a) The negation of the statement is that $\phi^{-1}(\{b\})$ contains at least two elements; show that this leads to a contradiction.

(b) Prove that for even n, ϕ_n is an injective function $Y \to X$; whereas for odd n, ϕ_n is an injective function $X \to X$.

(c) Apply part (a) to your result from part (b).

(d) Use induction!

Start with $\psi_1 = \psi_0 \circ g = f \circ g = f \circ \phi_0$. In the inductive step, you'd need to distinguish between even and odd values of n.

(e) To show that $\phi_{n-1}^{-1}(\{x\}) \subseteq \psi_n^{-1}(\{f(x)\})$ start with $z \in \phi_{n-1}^{-1}(\{x\})$ so that $\phi_{n-1}(z) = x$. Use part (d).

To show that $\psi_n^{-1}(\{f(x)\}) \subseteq \phi_{n-1}^{-1}(\{x\})$ start with $z \in \psi_n^{-1}(\{f(x)\})$ so that $\psi_n(z) = f(x)$. Use part (d) together with the injectivity of f

(f) To show that $f(X_{no}) \subseteq Y_{no}$, start with $x \in X_{no}$ and apply part (e) to conclude that for any $n \in \mathbb{N}$, $\psi_n^{-1}(\{f(x)\}) \neq \emptyset$. Extend to for any $n \in \mathbb{Z}_{\geq 0}$ by considering $n = 0$ separately.

To show that $Y_{no} \subseteq f(X_{no})$, start with $y \in Y_{no}$ and some $x \in X$ for which $f(x) = y$ (why must such an x exist?). Applying part (e) conclude that $x \in X_{no}$.

(g) The proof here is very similar to the previous part; try to make analogous arguments.

To show that $f(X_{even}) \subseteq Y_{odd}$, start with $x \in X_{even}$. Use part (e) to show that the smallest n for which $\psi_n^{-1}(\{f(x)\}) = \emptyset$ is $m + 1$, where m is the smallest for which $\phi_m^{-1}(\{x\}) = \emptyset$.

To show that $Y_{odd} \subseteq f(X_{even})$, start with $y \in Y_{odd}$ and some $x \in X$ such that $f(x) = y$ (why must such an x exist?). Show that the smallest n for which $\phi_n^{-1}(\{x\}) = \emptyset$ is one less than the smallest m for which $\psi_m^{-1}(\{y\}) = \emptyset$.

(h) This part is completely analogous to the previous part. Make sure you understand why and you can simply indicate the relevant substitutions.

(i) Define h as a piecewise function. Note that $X_{no}, X_{even}, X_{odd}$ partition X (and similarly $Y_{no}, Y_{even}, Y_{odd}$ partition Y).

(Exercise on page 158.)

Hint for Exercise 142 (Notation).

Distinguish between the preimage notation $f^{-1}(S)$ for a set S, and the inverse function notation $f^{-1}(y)$.

Does f have a well-defined inverse function? Recall that only bijective functions have a well-defined inverse.

Recall that $f^{-1}(S) := \{x \in X : f(x) \in S\}$ is the collection of all elements in the domain whose image under f is an element of the set S.

(Exercise on page 159.)

Hint for Exercise 143 (Images and Preimages)**.**

Recall that the image of a set S under the function f is $f(S) = \{f(x) : x \in S\}$, whereas the preimage of a set T under f is $f^{-1}(T) = \{x : f(x) \in T\}$.

One way of finding $f(S)$ is to apply the function rule to each element of S, computationally or symbolically.

For preimages, solve the condition $f(x) \in T$ for x; express the answer as the set of all such x. Pay attention to functions that may map several inputs to the same output.

(a)(iv) Recall that $\lceil x \rceil$ is the least integer greater or equal to x.

(b)(iv) Prove that f is surjective (in fact, it is even bijective).

(c)(iv) Note that $f^{-1}(\{-1\}) = \emptyset$, for example.

(Exercise on page 160.)

Hint for Exercise 144 (Preimages and Complements)**.**

Recall that S^c is the *complement* of S, i.e. $Y \setminus S$.

To prove two sets are equal, show each is contained in the other. Begin with $x \in f^{-1}(S^c)$ and carefully unpack what that means.

For example, to show that $f^{-1}(S^c) \subseteq (f^{-1}(S))^c$, explain each step in the sequence below

$$\begin{aligned} x \in f^{-1}(S^c) &\implies \\ f(x) \in S^c &\implies \\ f(x) \notin S &\implies \\ x \notin f^{-1}(S) &\implies \\ x \in (f^{-1}(S))^c. \end{aligned}$$

(Exercise on page 161.)

Hint for Exercise 145 (Images and Intersections).

(a) Start with an arbitrary $y \in f(A \cap B)$ and show that $y \in f(A) \cap f(B)$.

We have $\exists x \in A \cap B. f(x) = y$. Show that $f(x) \in f(A)$ and $f(x) \in f(B)$.

(b) Consider very simple functions first, then familiar functions, then finite functions, and so on.

For example, a constant function or the identity function. Some familiar algebraic functions.

(c) We need only ensure $f(A) \cap f(B) \subseteq f(A \cap B)$. Suppose $y \in f(A) \cap f(B)$; this means that $y \in f(A)$ and $y \in f(B)$. Therefore, there is some $a \in A$ such that $f(a) = y$, and also some $b \in B$ such that $f(b) = y$. We want to find $x \in A \cap B$ such that $f(x) = y$.

It would help if $a = b$; what condition on f guarantees that?

(d) Prove that if $f(A \cap B) = f(A) \cap f(B)$ for all $A, B \subseteq X$, then f is injective. Suppose $x_1, x_2 \in X$ are such that $f(x_1) = f(x_2)$, prove that $x_1 = x_2$.

Try $A = \{x_1\}$ and $B = \{x_2\}$ and apply the hypothesis on the function.

(Exercise on page 162.)

Hint for Exercise 146 (The Characterisitc Function).

(a) Suppose $x \in U$ is an arbitrary element, what is the value of $\chi_\emptyset(x)$? What about $\chi_U(x)$? Use the definitions of the characteristic function with S replaced with the appropriate set.

If $\emptyset \subsetneq S \subsetneq U$ then there are $x, y \in U$ such that $x \in S$ and $y \notin S$.

(b) For example, $\chi_S^{-1}(\{1\}) = \{x \in U \mid \chi_S(x) = 1\}$ (because $\chi_S(x) \in \{1\}$ is the same as saying $\chi_S(x) = 1$), what is this set?

Continuing with the same example, this set is simply $\{x \in U \mid x \in S\} = S$. Similar arguments apply to the other sets in the question (each with different results).

(c) Note $\chi_S(x) \in \{0, 1\}$; what are 0^2 and 1^2?

(d) The two functions have the same domain and codomain, so it remains to prove they have the same rule. This is the same as showing that they agree on the value of each $x \in U$. (Why is it the same?)

Let $x \in U$ be arbitrary, prove that $\chi_A(x) \cdot \chi_B(x) = 1$ if and only if $x \in A \cap B$.

(e) Similarly to the previous part, prove that $\chi_A(x)+\chi_B(x)-\chi_A(x)\chi_B(x)=1$ if and only if $x\in A\cap B$. We recommend to start by computing the function $\chi_A(x)+\chi_B(x)$, then use the previous part!

(f) Again, show that $1-\chi_A(x)=1$ if and only if $x\in A^c$ (and otherwise the value is 0).

(g) The idea is to compute this algebraically using the previous parts (though it is possible to compute it "from first principles", that is not what the question is trying to illustrate).

Starting from

$$A\triangle B=(A\cup B)\setminus(A\cap B)=(A\cup B)\cap(A\cap B)^c.$$

Denote $S:=A\cup B$ and $T=A\cap B$. Then, $A\triangle B=S\cap T^c$; use the results of the previous parts.

(Exercise on page 163.)

Hint for Exercise 147 (The Characteristic Function of $\mathbb{Z}$).

(a) Suppose x is an integer; what is the largest integer less than or equal to x?

(b) Note that the floor is always less than x; what about the ceiling?

(c) In part (a) you've found a sufficient condition; prove that it is also necessary.

If $\lfloor x \rfloor = \lceil x \rceil$, this common value must be x itself; why does this mean that x is an integer? (What is the definition of the floor function?)

(d) If $\lfloor x \rfloor < \lceil x \rceil$ then $\lfloor x \rfloor + 1 \leq \lceil x \rceil$ (why?). Prove the reverse inequality is also true.

Prove that $x \leq \lfloor x \rfloor + 1$ and conclude that $\lceil x \rceil \leq \lfloor x \rfloor + 1$.

(e) Study the (arithmetic) difference between the ceiling and the floor.

(Exercise on page 164.)

Hint for Exercise 148 (Functions, Preimages, and Partitions)**.**

(a) Suppose towards contradiction that there exists some $x \in f^{-1}(\{y_1\}) \cap f^{-1}(\{y_2\})$; what is $f(x)$? Why is this a contradiction?

(b) Show that the set on each side of the equality contains the other; one inclusion is directly from the definition of the preimage.

For the other inclusion, if $x \in X$, can you find some $y \in Y$ such that $x \in f^{-1}(\{y\})$?

(c) Recall that for a collection of subsets to form a partition they must satisfy *three* conditions!

Each subset must be nonempty; the subsets should be pairwise disjoint; and the subsets must cover the set. One of these three conditions fails in general.

To find a correct statement, for which $y \in Y$ can we guarantee that $f^{-1}(\{y\}) \neq \emptyset$?

(d) Note that the collection of preimages does partition $\mathbb{R}$ in this case.

To determine the block $f^{-1}(\{n\})$, find $x \in \mathbb{R}$ such that $f(x) = \lfloor x \rfloor = n$.

It may help to revisit your sketch of the graph of the floor function (Exercise 8 in the Introduction to Functions handout).

(Exercise on page 165.)

Part III

Answers

Solution for Exercise 1 (Vocabulary)**.**

Let us address each symbol in turn:

- The symbol $:=$ is used to indicate that the expression appearing on the right-hand side is the *definition* of the symbol appearing on the left-hand side.

  An example of **correct** usage is
  $$S := \left\{n^2 \,:\, n \in \mathbb{Z}\right\}.$$
  The expression above could be read as "Let S be the set of (integral) squares" or "Let S denote the set of squares".[1]

  An example of an **incorrect** usage is: let x be the unique positive root of $x^2 + 3x - 10$, then $x := 5$.

  The *definition* of the variable x is "the unique positive root of $x^2 + 3x - 10$". We can calculate the value of x and find (correctly) that $x = 5$, but that is an "additional fact" about this variable x, not its definition. (On the other hand, the fact follows from the definition.)

  Here is a similar example, which again contrasts the uses of the equality symbol $=$ and the definition symbol $:=$
  $$A := \{2\},$$
  $$B := \{p \in \mathbb{N} \,:\, p \text{ is even AND } p \text{ is prime}\}.$$
  We can then correctly write $A = B$ (but it would be **incorrect** to write $A := B$).

- The set-membership symbol $\in$ is used to indicate that the symbol appearing on the left-hand side is an *element of* the set appearing on the right-hand side.

  Examples of **correct** usage include
  $$1 \in \mathbb{N}, \qquad -3 \in \mathbb{Z}, \qquad \pi \in \mathbb{R}, \qquad \sqrt{2} \notin \mathbb{Z}.$$
  The symbol $\notin$ as in the last expression above ($\sqrt{2} \notin \mathbb{Z}$) is used to say that what appears on the left-hand side ($\sqrt{2}$) is *not an element of* the set appearing on the right-hand side ($\mathbb{Z}$).[2]

  Examples of **incorrect** usage[3] include
  $$\mathbb{N} \in \mathbb{Z}, \qquad \mathbb{N} \notin \mathbb{Z}, \qquad \mathbb{N} \in 1, \qquad 1 \in 0.$$
  If you have seen the subset-symbol $\subseteq$ before, it is worth contrasting it with the set-membership symbol; this is a common point of confusion.

- In the context of Section 2.1, the divisibility symbol $\mid$ indicates that the number on the left-hand side divides the number on the right-hand side.

[1] Less formally, we could also write
$$S := \{1^2, 2^2, 3^2, \ldots\}$$
but it is preferable to use set-builder notation whenever possible because it is more accurate (the reader does not need to guess the pattern the author intended).

[2] Note that $\sqrt{2} \in \mathbb{Z}$ is an example of correct usage, but the statement it asserts is mathematically incorrect! "Correct usage" merely means the statement is mathematically grammatical (in computer language: it compiles); it can still be a nonsensical grammatical statement.

[3] In the context of our course.

Examples of **correct** usage include

$$3|6, \qquad -11|121, \qquad a|2ab, \qquad b|b^3.$$

In addition, $5|3$ is an example of correct usage but an incorrect statement! Just as with the set-membership symbol, we can write $5 \nmid 3$ for the statement "5 does not divide 3" (this would be both a correct usage and a correct statement. Can you give an example of a correct usage but incorrect statement for the symbol $\nmid$?)

Examples of **incorrect** usage include

$$0.5|1, \qquad 4|\{4, 8, 12, \dots\}, \qquad 1|\sqrt{2}.$$

Note that we have only defined the divisibility symbol when there are integers on each side of it. (Can you find a better way of expressing the middle statement above about divisibility by 4?)

- Each of the symbols $\mathbb{N}, \mathbb{Z}, \mathbb{R}$ stands for a particular set:
  - $\mathbb{N}$ denotes the set of natural numbers. Examples of **correct** usage include

    $$1 \in \mathbb{N}, \qquad 5 \in \mathbb{N}, \qquad \left\{a^2 : a \in \mathbb{N}\right\}, \qquad \{a \in \mathbb{Z} : a > 0\} = \mathbb{N}.$$

    Note that each of $0 \in \mathbb{N}$, $-5 \in \mathbb{N}$, and $\mathbb{N} = \mathbb{Z}$ is an example of correct usage but an *incorrect* statements.

    Examples of **incorrect** usage include

    $$\mathbb{N} \in 1, \qquad \mathbb{N} > \{1, 2\}, \qquad 1|\mathbb{N}, \qquad \mathbb{N} \in \mathbb{Z}.$$

  - $\mathbb{Z}$ denotes the set of integers (also known as "whole numbers")[4]. Examples of **correct** usage include

    $$-1 \in \mathbb{Z}, \qquad 0 \in \mathbb{Z}, \qquad \left\{a^2 : a \in \mathbb{Z}\right\}, \qquad \{a \in \mathbb{Z} : a > 0\} = \mathbb{N}.$$

    Note that each of $\frac{3}{5} \in \mathbb{Z}$, $-\sqrt{2} \in \mathbb{Z}$, and $\mathbb{N} = \mathbb{Z}$ is an example of correct usage but an *incorrect* statement.

    Examples of **incorrect** usage include

    $$\mathbb{N} \in \mathbb{Z}, \qquad \{-1, -2, -3, \dots\} \in \mathbb{Z}, \qquad \mathbb{Z} > \mathbb{N}.$$

  - $\mathbb{R}$ denotes the set of real numbers (including rational and irrational numbers[5].). Examples of **correct** usage include

    $$0 \in \mathbb{R}, \qquad -\pi \in \mathbb{R}, \qquad \left\{a^2 : a \in \mathbb{R}\right\} = \{a \in \mathbb{R} : a \geq 0\}, \qquad \mathbb{Z} \neq \mathbb{R}.$$

    Note that each of $\mathbb{R} = \mathbb{Z}$ and $\{a^2 : a \in \mathbb{R}\} = \mathbb{R}$ is an example of correct usage but an *incorrect* statement.

    Examples of **incorrect** usage include

    $$\mathbb{Z} \in \mathbb{R}, \qquad \mathbb{R} > \mathbb{N}, \qquad \mathbb{R} \in \sqrt{2}, \qquad \mathbb{R} = \pi.$$

(Exercise on page 11.)

[4] The symbol originates in the first letter of the German word "Zahlen" meaning "numbers".

[5] Informally, the real numbers include all possible decimal expansions, such as 123.345947957...

Solution for Exercise 2 (Parity).

> **Definition 2.1.** An integer n is **even** if $n = 2k$ for some $k \in \mathbb{Z}$. An integer n is **odd** if $n = 2k + 1$ for some $k \in Z$.

(a) The number 246 is even, since $246 = 2 \cdot 123$ and $123 \in \mathbb{Z}$.

The number 101 is odd, since $101 = 2 \cdot 50 + 1$ and $50 \in \mathbb{Z}$.

(b) To prove that 101 is *not even* we must show that it is *impossible* to satisfy the definition. That is, we must prove that there is no $k \in Z$ such that $101 = 2k$. Equivalently, we must prove that for *every possible* $k \in \mathbb{Z}$ *we have* $101 \neq 2k$

Since $\mathbb{Z}$ is infinite, we would never be able to prove this just by showing examples of $k \in \mathbb{Z}$ such that $101 \neq 2k$. We need more tools!

The text gave us the following very useful *assumption*[6]

> "For the remainder of this section, you may assume that every integer is either even or odd but never both."

Using this assumption, we can prove that 101 is not even just by showing it is odd; which we have already done in part (a) above![7]

(Exercise on page 12.)

[6] We call it an "assumption" because we haven't proved it. However, it is possible to prove this statement, and one does so in a Number Theory course such as MAT 315.

[7] There are other ways of proving that 101 is not even, but they all require more assumptions or tools. For example, if we know some facts about the *ordering* of integers, we can prove that for $k \leq 50$ we have $101 > 2k$, while for $k > 50$ we have $101 < 2k$; and this exhausts all possible integers. (We will discuss proofs by exhaustion later in the course.) There are often many different ways of proving a correct statement (there are *hundreds* known proofs of the Pythagorean theorem, for example). The key point to keep in mind is to always be clear what *definitions*, *assumptions*, *logical steps*, and *theorems/facts* the proof is using.

Solution for Exercise 3 (Squaring).

Suppose n is an even integer. By Definition 2.1 this means that $n = 2k$ for some $k \in \mathbb{Z}$. Therefore, $n^2 = 4k^2 = 2(2k^2)$. Since $2k^2 \in \mathbb{Z}$ we have shown that $n^2 = 2j$ for some $j \in \mathbb{Z}$ (namely, for $j = 2k^2$). By Definition 2.1 this means that n^2 is an even integer.

(a) We have used Definition 2.1 twice in the proof. First, to "unpack"[8] the *hypothesis*[9] that n is an even integer. Then to show that n^2 also satisfies the definition of an even integer.

(b) Apart from Definition 2.1 we have made one more assumption: that if $k \in \mathbb{Z}$ then $2k^2 \in \mathbb{Z}$. We are using the facts that we can add and multiply integers and the result is still an integer.[10]

Helpful Tip!

Note that in order to prove that n^2 is even we showed that $n = 2j$ for some $j \in \mathbb{Z}$. This is often a point of confusion, because Definition 2.1 is phrased in terms of k.

The k appearing in Definition 2.1 is called a *dummy variable*. The following definition is completely equivalent to Definition 2.1:

> **Definition 2.1.** An integer n is **even** if $n = 2j$ for some $j \in \mathbb{Z}$. An integer n is **odd** if $n = 2r + 1$ for some $r \in Z$.

The word "some" in the phrase "for some $j \in \mathbb{Z}$" is the reason this variable is a dummy variable. We will discuss this more when we talk about *quantification* in the next couple of weeks.

If we tried to phrase the first part Definition 2.1 in natural language we could say

> An integer is even if it is twice another integer.

This might help the reader see that no particular integer is specified (just "another integer").[a] In our proof, we have shown that $n^2 = 4k^2$ which is indeed "twice another integer"; it is twice the integer $2k^2$.

[a] However, such phrasing becomes very confusing very fast, which is the reason humans have started using symbols in mathematics.

(Exercise on page 13.)

[8] or "explain" or "translate"

[9] or "data" or "givens" or "assumptions"

[10] These facts are not obvious; for example, if we divide two integers the result is not necessarily an integer. It is possible to prove these facts, and courses which develop numbers axiomatically (such as Abstract Algebra or Set Theory or Mathematical Logic) may do so. Our course focuses more on proof *techniques* rather than axiomatic development of mathematics.

Solution for Exercise 4 (Divisibility)**.**

Here are the two relevant definitions (using different variable names, solely for convenience):

> **Definition 2.1.** An integer n is **even** if $n = 2k$ for some $k \in \mathbb{Z}$. An integer n is **odd** if $n = 2k + 1$ for some $k \in Z$.

> **Definition 2.5.** Given $a, b \in \mathbb{Z}$, we say that a **divides** b, written $a|b$, if there exists $r \in \mathbb{Z}$ such that $b = ar$. If $a|b$, we may also say that b is **divisible by** a or that a is a **factor** of b.

(a) Suppose n is an even integer. By Definition 2.1, $n = 2k$ for some $k \in \mathbb{Z}$. By Definition 2.5, $2|n$ means that there exists $r \in \mathbb{Z}$ such that $n = 2r$. Since $n = 2k$ and $k \in Z$, we may take $r = k$, which proves that $2|n$.

(b) Suppose n is divisible by 2. By Definition 2.5, this means there exists some $r \in \mathbb{Z}$ such that $n = 2r$. By Definition 2.1, this means that n is even (we may take $k = r$ in that definition).

(c) The statements in parts (a) and (b) are each the **converse** of the other. As logical statements, they are distinct and their proofs are different. For proving the first statement we started with the assumption that n is even, whereas in the second statement we are trying to prove that n is even.

We cannot use one statement to prove the other (without going back to the definitions). Abstractly, statement (i) and (ii) are of the form "if A then B" and "if B then A". In our case, A is "n is even", while B is "n is divisible by 2". In other cases, A may be "the shape is a square", while B may be "the shape is a rectangle"; in such a case it is clear that just because "if a shape is a square then it is a rectangle" is true does not mean that the converse "if a shape is a rectangle then it is a square" is true!

(d) To prove that n is *not* divisible by 2 we must show there does not exist some $r \in \mathbb{Z}$ such that $n = 2r$. In other words, we need to show that for every $r \in \mathbb{Z}$ we have $n \neq 2r$.

(e) According the assumption in the text

> "For the remainder of this section, you may assume that every integer is either even or odd but never both."

If n is an odd integer, we know it is *not* even. On the other hand, if n is divisible by 2 then we have shown in part (b) above that it is even. Since a number cannot be both even and not even, if n is odd it cannot be divisible by 2.

This is an example of a *proof by contradiction*, a proof technique which we will discuss in detail later in the course.

(Exercise on page 14.)

Solution for Exercise 5 (Translation)**.**

(a) If it is currently raining in Toronto, then Mai the Mathematician is holding an umbrella. We can express this proposition as $R \implies U$.

(b) It is not currently raining in Toronto. We can express this proposition as $\neg R$.

(c) It is currently raining in Toronto or Mai the Mathematician is holding an umbrella. We can express this proposition as $R \vee U$.

(d) Mai the Mathematician is holding an umbrella if and only if it is currently raining in Toronto. We can express this proposition as $R \iff U$.

(e) It is currently raining in Toronto and Mai the Mathematician is holding an umbrella. We can express this proposition as $R \wedge U$.

(f) Whenever Mai the Mathematician is not holding in umbrella, it is not raining in Toronto. We can express this proposition as $(\neg U) \implies (\neg R)$. Note that this is the contrapositive of $R \implies U$; the two statements are logically equivalent. (Do you see why?)

(Exercise on page 15.)

Solution for Exercise 6 (Truth Values).

(a) $R \implies U$.

- The only situation in which this proposition is false is when it is raining in Toronto and yet Mai the Mathematician does not hold an umbrella.
- An example for a situation in which the proposition is true (vacuously) is when it is *not* currently raining in Toronto, regardless of whether or not Mai the Mathematician is holding an umbrella!

Helpful Tip!
Recall this important distinction between the mathematical implication and the everyday usage of "if, then". Suppose Mai the Mathematician entered into a contract which says "Whenever it is currently raining in Toronto, (then) Mai the Mathematician is holding an umbrella." When it is not raining in Toronto, there is absolutely no way that Mai is violating the contract.

(b) $\neg R$.

- The only situation in which this proposition is false is when it is currently raining in Toronto.
- The only situation in which this proposition is true is when it is not currently raining in Toronto.

(c) $R \vee U$.

- The only situation in which this proposition is false is when it is not raining in Toronto *and* Mai the Mathematician is not holding an umbrella[11].
- An example for a situation in which the proposition is true is when it is currently raining in Toronto (regardless of whether or not Mai the Mathematician is holding an umbrella).

(d) $R \iff U$.

- An example for a situation in which the proposition is false is when it is currently raining in Toronto and Mai the Mathematician is not holding an umbrella. Another example is when it is not currently raining in Toronto and yet Mai the Mathematician is holding an umbrella.
- An example for a situation in which the proposition is true is when it is both currently raining in Toronto and Mai the Mathematician is holding an umbrella. Another example is when (both) it is not currently raining in Toronto and Mai the Mathematician is not holding an umbrella.

(e) $R \wedge U$.

- An example for a situation in which the proposition is false is when Mai the Mathematician is holding an umbrella even though it is not currently raining in Toronto.
- The only situation in which the proposition is true is when it is raining in Toronto and Mai the Mathematician is holding an umbrella.

[11] One would imagine this happens most of the time!

(f) $(\neg U) \implies (\neg R)$. As we remarked before, this proposition is logically equivalent to $R \implies U$ and so, for every situation in which $R \implies U$ is true/false it will also be the case that $(\neg U) \implies (\neg R)$ is true/false. In particular, the only situation in which $(\neg U) \implies (\neg R)$ is false is when it is raining in Toronto and yet Mai the Mathematician does not hold an umbrella.

(Exercise on page 16.)

Solution for Exercise 7 (Truth Tables).

Here are the filled truth-tables; syntactically, these could be taken as the *definition* of each connective.

A	$\neg A$
0	1
1	0

A	B	$A \wedge B$
0	0	0
0	1	0
1	0	0
1	1	1

A	B	$A \vee B$
0	0	0
0	1	1
1	0	1
1	1	1

A	B	$A \implies B$
0	0	1
0	1	1
1	0	0
1	1	1

A	B	$A \iff B$
0	0	1
0	1	0
1	0	0
1	1	1

A	B	$\neg A$	$(\neg A) \vee B$	$A \implies B$
0	0	1	1	1
0	1	1	1	1
1	0	0	0	0
1	1	0	1	1

Since the columns for $(\neg A) \vee B$ and $A \implies B$ are identical, this is a proof that these two propositions are logically equivalent.

A	B	$A \implies B$	$\neg(A \implies B)$	$\neg B$	$A \wedge (\neg B)$
0	0	1	0	1	0
0	1	1	0	0	0
1	0	0	1	1	1
1	1	1	0	0	0

Since the columns for $\neg(A \implies B)$ and $A \wedge (\neg B)$ are identical, this is a proof that these two propositions are logically equivalent.

A	B	$A \implies B$	$B \implies A$	$(A \implies B) \wedge (B \implies A)$	$A \iff B$
0	0	1	1	1	1
0	1	1	0	0	0
1	0	0	1	0	0
1	1	1	1	1	1

Since the columns for $(A \implies B) \wedge (B \implies A)$ and $A \iff B$ are identical, this is a proof that these two propositions are logically equivalent.

(Exercise on page 17.)

Solution for Exercise 8 (Arithmetic)**.**

(a) $A \wedge B$ can be represented by $A \cdot B$. That is, for conjunction we multiply the values of A and B.

(b) $A \vee B$ can be represented by $A + B - AB$. It is possible to "guess" this expression, but another way to derive it is from the logically equivalent expression $\neg(\neg A \wedge \neg B)$ (an example of one of *DeMorgan's Laws*) so that our previous observations imply $A \vee B = 1 - (1 - A)(1 - B)$.

(c) $A \implies B$ can be represented by $1 - A + AB$. The easiest way to find this expression is to use the logically equivalent form of the implication $(\neg A) \vee B$ and out previous observations to find $(1 - A) + B - (1 - A)B$.

(d) $A \iff B$ can be represented by $(1-A+AB)(1-B+AB)$ (other equivalent expressions are also possible). The easiest way to derive this is to use the logically equivalent form of the biconditional $(A \implies B) \wedge (B \implies A)$ and our previous observations.

(Exercise on page 18.)

Solution for Exercise 9 (Propositions)**.**

(a) The integer 2 is an even number and a prime number. *True*.

(b) The integer 2 is an even number or a prime number. *True*.

(c) If the integer 2 is an even number, then it is an odd number. *False*.[12]

(d) If the integer 2 is an odd number, then it is an even number. *True* (vacuously).

(e) If the integer 2 is a prime number, then it is even. *True* (trivially).

(f) The integer 2 is an even number if and only if it is a prime number. *True*.

(g) The integer 2 is an odd number if and only if it is a prime number. *False*.

(h) The integer 4 is an odd number if and only if it is a prime number. *True*.

(Exercise on page 19.)

[12]We are using the fact that "not even" means "odd". We may do so under the assumption that each integer is either even or odd (but not both), which is one of the facts introduced in §2.1.

Solution for Exercise 10 (Cards).

This exercise is a version of the so-called *Wason selection task.* If we let E be the proposition “one side is an even number” and V be the proposition “one side is a vowel”, the hypothesis can be translated to the logical statement

$$E \implies V.$$

To check this statement we only need to check that every time E is true, V is also true. Therefore, only the cards showing 2 must be turned over. If the other side contains a vowel, the statement is true; otherwise, the statement is false.

Make sure you can fully explain for each remaining card why it does not need to be turned over; this understanding will help you in the next sections when we discuss proof techniques for conditional statements.

(Exercise on page 20.)

Solution for Exercise 11 (Nested)**.**

(a) We can see directly from the truth-table that $A \implies B$ is false only when $A = T$ and $B = F$.

A	B	$A \implies B$
T	T	T
T	F	F
F	T	T
F	F	T

(b) Here the roles of A and B are swapped. The proposition $B \implies A$ (the *converse* of $A \implies B$) is false only when $A = F$ and $B = T$.

(c) The proposition $A \implies (B \implies C)$ is of the form $\alpha \implies \beta$ and is false only when $\alpha = T$ and $\beta = F$. Since α is just A, we see that $A = T$. Next, β is the proposition $B \implies C$ and since $\beta = F$ we must have $B = T$ and $C = F$. In summary, $A = B = T$ and $C = F$.

(d) The proposition $(A \implies B) \implies C$ is of the form $\alpha \implies \beta$ and is false only when $\alpha = T$ and $\beta = F$. Since β is just C, we see that $C = F$. On the other hand, α is the proposition $A \implies B$ and there are many truth-value assignments to A, B which make this proposition true! Therefore, we cannot determine the truth-value of A, B; we can only affirm that it is not the case that $A = T$ and $B = F$ (but any other combination for A, B is possible).

(e) Since $\alpha \implies \beta$ is false if and only if α is true and β is false, we may conclude that A is true and

$$B \implies (C \implies (\cdots(Y \implies Z)\cdots))$$

is false. By the same reasoning, B is true and

$$C \implies (\cdots(Y \implies Z)\cdots)$$

is false. Continuing in this manner we conclude that $A, B, \ldots, Y$ are all true and Z is false.[13]

Remark: Note that if we were given that the long $A \implies \cdots$ statement is true instead of false, we would not have been able to determine the true value of even a single one of the propositions $A, \ldots, Z$. (Why not?)

(Exercise on page 21.)

[13] This is an example of reasoning by *induction* (or *recursion*), which we shall discuss at greater detail later in the course.

Solution for Exercise 12 (Words)**.**

(a) Let us address each term in turn:

- The *converse* is the proposition $B \implies A$. (Note that the conjunction of the implication and its converse form the biconditional.)
- The *inverse* is the proposition formed by negating each component: $(\neg A) \implies (\neg B)$.
- The *contrapositive* is $(\neg B) \implies (\neg A)$ and is the only one of the three which is logically equivalent to the original implication $A \implies B$.

(b) Use logical connectives to express each of the following word-descriptions:

(i) A is necessary for B. This means that whenever B is true, A must be true, which we can express as $B \implies A$.

(ii) A is sufficient for B. This means that whenever A is true, B must be true, which we can express as $A \implies B$.

(iii) A only if B. This means that whenever A is true, B must be true, which we can express as $A \implies B$.

(iv) A if B. This means that whenever B is true, A must be true, which we can express as $B \implies A$.

(v) A whenever B. This means that whenever B is true, A must be true, which we can express as $B \implies A$.

(Exercise on page 22.)

Solution for Exercise 13 (Equivalence).

Implication $A \implies B$ is equivalent to $(\neg B) \implies (\neg A)$ (its contrapositive) and also to $\neg(A \wedge (\neg B))$ (based on its truth-table), which by DeMorgan's Laws is equivalent to $(\neg A) \vee B$. Therefore, (a), (d), (h) are logically equivalent. The same reasoning shows that (b), (c), (i) are also logically equivalent.

Implication negation Since $A \implies B$ is equivalent to $(\neg A) \vee B$, its negation $\neg(A \implies B)$ is equivalent to $A \wedge (\neg B)$. That is, (e) and (k) are equivalent.

DeMorgan's Laws state that $\neg(A \wedge B)$ is logically equivalent to $(\neg A) \vee (\neg B)$ and dually $\neg(A \vee B)$ is logically equivalent to $(\neg A) \wedge (\neg B)$. Therefore, (f) is equivalent to (m); while (g) is equivalent to (l).

(Exercise on page 23.)

Solution for Exercise 14 (Complete Set of Connectives)**.**

(a) Using only $\neg, \vee$ we need to re-express three of the five compound propositions:

- $A \wedge B$ can be re-expressed using DeMorgan's Laws $\neg((\neg A) \vee (\neg B))$;
- $A \implies B$ is logically equivalent to $(\neg A) \vee B$;
- $A \iff B$ is logically equivalent to $(A \implies B) \wedge (B \implies A)$ which we can rewrite as $((\neg A) \vee B)) \wedge ((\neg B) \vee A)$. Then, using DeMorgan's Laws we obtain

$$\neg\left((\neg((\neg A) \vee B))) \vee (\neg((\neg B) \vee A))\right).$$

(b) Using only $\neg, \wedge$ we need to re-express three of the five compound propositions. One way to approach this part is to apply DeMorgan's Laws to the previous part.

- $A \vee B$ by DeMorgan's Laws is equivalent to $\neg((\neg A) \wedge (\neg B))$;
- $A \implies B$ is equivalent to $\neg(\neg(A \implies B))$ which is equivalent to $\neg(A \wedge (\neg B))$.
- $A \iff B$ is equivalent to $(A \implies B) \wedge (B \implies A)$ which, using the previous part, is equivalent to
$$(\neg(A \wedge (\neg B))) \wedge (\neg(B \wedge (\neg A))).$$

(c) Using only $\neg, \implies$ we need to re-express three of the five compound propositions.

- $A \vee B$ is equivalent to $(\neg(\neg A)) \vee B$ which is equivalent to $(\neg A) \implies B$;
- $A \wedge B$ is equivalent to $A \wedge (\neg(\neg B))$ which is equivalent to $\neg(A \implies (\neg B))$;
- $A \iff B$ is equivalent to $(A \implies B) \wedge (B \implies A)$ which, using the previous part, is equivalent to
$$\neg((A \implies B) \implies (\neg(B \implies A)).$$

(d) The first key observation is that $\neg A$ is logically equivalent to $A \uparrow A$. The second key observation is that the truth-table of $A \uparrow B$ is the mirror-image of $A \wedge B$.

A	B	$A \uparrow B$	$A \wedge B$
0	0	1	0
0	1	1	0
1	0	1	0
1	1	0	1

That is $A \wedge B$ is logically equivalent to $\neg(A \uparrow B)$ which, using the first key observation, is logically equivalent to $(A \uparrow B) \uparrow (A \uparrow B)$. Now we know how to express $\neg$ and $\wedge$, we can just plug the expressions into part (b) above.

Remark: The connective $\uparrow$ is known as *NAND* (it is the negation of "and" so "not and"). Because it forms a complete set of connectives by itself, it is commonly implemented in electronic circuits.

(Exercise on page 24.)

Solution for Exercise 15 (Tautologies and Contradictions).

(a) $((A \wedge B) \implies C) \implies (A \implies (B \implies C))$ is a tautology, it is true regardless of the truth values of A, B, and C. To see this, we shall show that the proposition is never false. Because an implication $\alpha \implies \beta$ is false if and only if $\alpha = T$ and $\beta = F$, suffice it to show that $\alpha = F$ whenever $\beta = F$.[14] In our case, β is the compound proposition $A \implies (B \implies C)$ which is false just in case $A = B = T$ and $C = F$ (see also Exercise 11). With those truth values, $A \wedge B = T$ and $C = F$ so that α, which is $(A \wedge B) \implies C$, is false.

(b) $((\neg A) \wedge B) \implies ((\neg B) \vee C)$ is neither a tautology nor a contradiction, which we can demonstrate by finding truth-value assignments for A, B, C which make the proposition true and other truth-value assignments for A, B, C which make the proposition false.

- If we choose $B = F$ then the antecedent $((\neg A) \wedge B)$ is false, so the implication is (vacuously) true.
- If we choose $A = F$, $B = T$, and $C = F$ then the antecedent $((\neg A) \wedge B)$ is true, but the consequent $((\neg B) \vee C)$ is false, so the implication is false.

(c) $(A \implies (B \implies C)) \wedge ((A \wedge B \wedge (\neg C))$ is a contradiction. This proposition is of the form $\alpha \wedge \beta$, so to show it is never true, it is enough to show that when $\beta = T$ we must have $\alpha = F$. In our case, β is the proposition $(A \wedge B \wedge (\neg C)$ which is true only when $A = B = T$ and $C = F$; but with these truth-value assignments, the proposition $A \implies (B \implies C)$ (which is α) is false (see Exercise 11).

(Exercise on page 25.)

[14] It is worth pausing to think about this point; this is exactly the contrapositive form.

Solution for Exercise 16 (Contrapositive Statements)**.**

(a) If $x + 2 \leq 5$ then $x \leq 3$.

(b) If tomorrow is not Thursday, then today is not Wednesday.

(c) If f is not continuous at x, then f is not differentiable at x.

(d) If n is not a multiple of 3, then n is not a multiple of 6.

Each one of the original assertions is a true proposition. The contrapositive form is logically equivalent to the original form, so each of the statements above is also true!

(Exercise on page 27.)

Solution for Exercise 17 (Direct Proofs)**.**

(a) Assume a and b are integers such that $a \mid b$, then $b = ak$ for some $k \in \mathbb{Z}$, by the definition of divisibility. Then $bc = (ak)c = a(kc)$, so by the definition of divisibility, $a \mid bc$.

(b) Assume m and n are odd integers, then $m = 2k + 1$, and $n = 2\ell + 1$ for some integers k and ℓ. Then

$$m + n = (2k + 1) + (2\ell + 1) = 2k + 2\ell + 2 = 2(k + \ell + 1).$$

We can conclude that $m + n$ is in fact even.

> **Helpful Tip!**
> Note that we had to choose different symbols k and ℓ when instantiating $m = 2k + 1$ and $n = 2\ell + 1$. This is because m and n are potentially two different numbers (though they may be the same, we just don't know)! If we had $m = 2k+1$ and $n = 2k+1$ we are in fact asserting that $m = n$.

(c) Assume $a \mid b$ and $b \mid c$. Then $b = ak$ and $c = b\ell$ for some integers k and ℓ (by the definition of divisibility). But then, $c = b\ell = (ak)\ell = a(k\ell)$, so (by the definition of divisibility) $a \mid c$.

(d) Assume n is divisible by 6. So for some integer k, we have $n = 6k$. Then $n = 2(3k)$ so n is divisible by 2. Moreover, $n = 3(2k)$, so n is divisible by 3.

(Exercise on page 28.)

Solution for Exercise 18 (Contra- Proofs)**.**

(a) The contrapositive form is: if n is odd, then n^2 is odd.

> **Helpful Tip!**
> Here we are using an important assumption from the textbook:
>
> "...you may assume that every integer is either even or odd but never both."
>
> This means that the negation of "even" is "odd" and vice versa.

Assume n is odd, then $n = 2k + 1$ for some integer k. We have

$$n^2 = (2k + 1)^2 = 2(2k^2 + 2k) + 1.$$

Therefore, n^2 is also odd. This proves the original statement.

For a proof by contradiction: assume n^2 is even and suppose towards contradiction that n is odd. Then $n = 2k + 1$ for some $k \in \mathbb{Z}$ and $n^2 = (2k + 1)^2 = 2(2k^2 + 2k) + 1$ is odd. Since n^2 was assumed to be even, this is a contradiction. This contradiction proves that if n^2 is even, then n is even.

(b) The contrapositive form is: if n is divisible by 3, then n^2 is divisible by 3. Indeed, if $3|n$ then $n = 3k$ for some $k \in \mathbb{Z}$ so that $n^2 = 9k^2 = 3(3k^2)$ and so $3|n^2$.

For a proof by contradiction: assume n^2 is not divisible by 3 and suppose toward contradiction that n is divisible by 3. Then $n = 3k$ for some $k \in \mathbb{Z}$ so that $n^2 = 9k^2 = 3(3k^2)$ showing that n^2 is divisible by 3. Since n^2 was assumed to be not divisible by 3, this is a contradiction. This contradiction proves that if n^2 is not divisible by 3 then n is not divisible by 3.

(c) The contrapositive form is: if n is even then $7n^3$ is even. Indeed, if n is even then $n = 2k$ for some $k \in \mathbb{Z}$ so that $7n^3 = 7 \cdot 8k^3 = 2(28k^3)$ so that $7n^3$ is even.

For a proof by contradiction: assume $7n^3$ is odd and suppose toward contradiction that n is even. Then $n = 2k$ for some $k \in \mathbb{Z}$ so that $7n^3 = 2(28n^3)$ so that $7n^3$ is even. Since $7n^3$ was assumed to be odd, this is a contradiction.

(d) The contrapositive form is: suppose none of m, n is even, then $m \cdot n$ is not even. Indeed, if m, n are odd, then $m = 2k + 1$ and $n = 2\ell + 1$ for some $k, \ell \in \mathbb{Z}$. Then, $m \cdot n = (2k + 1)(2\ell + 1) = 2(2k\ell + k + \ell) + 1$ is odd.

For a proof by contradiction: assume $m \cdot n$ is even and suppose towards contradiction that neither of m, n is even. Then each of m, n is odd, so $m = 2k + 1$ and $n = 2\ell + 1$ for some $k, \ell \in \mathbb{Z}$. Then, $m \cdot n = 2(2k\ell + k + \ell) + 1$ is odd, contradicting the assumption that it is even.

$$n^2 = (3k + 2)^2 = 9k^2 + 12k + 4 = 3(3k^2 + 4k + 1) + 1,$$

(Exercise on page 29.)

Solution for Exercise 19 (Direct Proofs II)**.**

(a) First solution: Suppose $x > 3$. In particular, x is positive and we may multiply both sides of the inequality by x to conclude $x^2 > 3x$. Similarly, we may multiply both sides of $x > 3$ by 3 to conclude that $3x > 9$. Therefore, $x^2 > 3x > 9$ so that $x^2 > 9$.

Second solution: Suppose $x > 3$, then $x - 3 > 0$, and $x + 3 > 6 > 0$. Multiply both sides of $x + 3 > 0$ by $x - 3$, which is positive, to get $(x - 3)(x + 3) > 0$ so $x^2 - 9 > 0$ and $x^2 > 9$.

(b) First solution: Suppose $0 < x < y$. Multiply both sides of $x < y$ by x to conclude $x^2 < xy$. Multiply both sides of $x < y$ by y to conclude $xy < y^2$. Therefore, $x^2 < xy < y^2$ so that $x^2 < y^2$.

Second solution: If $0 < x < y$, then x and y are both positive numbers. Also, $y - x$ and $y + x$ are both positive. So $(y - x)(y + x) > 0$ so that $y^2 - x^2 > 0$ which gives us $y^2 > x^2$.

(c) First solution: Suppose $0 \leq x \leq 1$. Multiply both sides of the inequality $x \leq 1$ by x to conclude $x^2 \leq x$.

Second solution: If $0 \leq x \leq 1$, then $x \geq 0$ and $x - 1 \geq 0$, so that $x(1 - x) \geq 0$. It follows that $x - x^2 \geq 0$, and so $x^2 \leq x$.

(d) If $0 < x < y$, Then $y - x > 0$ and xy is positive. Dividing $y - x > 0$ by xy gives $\frac{y}{xy} - \frac{x}{xy} > \frac{0}{xy}$, which reduces to $\frac{1}{x} - \frac{1}{y} > 0$, and so $\frac{1}{x} > \frac{1}{y}$.

(e) If $|x| < 1$, then $0 < |x| < 1$. By part (b), $|x|^2 < 1^2$. But $|x|^2 = x^2$ and so $x^2 < 1$.

(Exercise on page 30.)

Solution for Exercise 20 (Contra- Proofs II).

(a) The contrapositive is: if $a = b$, then $a^2 = b^2$. This is immediate. Therefore if $a^2 \neq b^2$, then $a \neq b$.

For a proof by contradiction: assume that $a^2 \neq b^2$ and suppose that $a = b$, then $a^2 = b^2$, a contradiction.

(b) The contrapositive is: if $x > 1 + x^2$ then $x < 0$. Indeed, suppose $x > 1 + x^2$ so that $x^2 - x + 1 < 0$. Now, $x^2 - x + 1$ is the equation of a parabola with minimum at $x = 1/2$. When $x = 1/2$ then $x^2 - x + 1 = 3/4 > 0$, so we see that the function $x^2 - x + 1$ is never negative. In particular, the condition "if $x > 1 + x^2$" is always false, so the implication is always (vacuously) true!

For a proof by contradiction: assume $x \geq 0$ and suppose for the sake of contradiction that $x > 1+x^2$ or, equivalently, $x^2 - x + 1 < 0$. Now $x^2 - x + \frac{1}{4} = (x - \frac{1}{2})^2 \geq 0$, so $x^2 - x + 1 = (x - \frac{1}{2})^2 + \frac{3}{4} > 0$, a contradiction.

Helpful Tip!

Note that in our proof by contradiction we haven't used the assumption $x \geq 0$ at all. This is to be expected, since our proof by contrapositive shows that $x \leq 1 + x^2$ is always true. In other words, we are proving an implication of the form $A \implies B$ with B being true.

(c) The contrapositive is: if $x^2 \leq 25$ then $|x| \leq 5$. Indeed, suppose $x^2 \leq 25$ so that $x^2 - 25 < 0$ which we can rewrite as $(x-5)(x+5) < 0$. The product is negative if and only exactly one of the multiplicands is negative, so if and only if $-5 \leq x \leq 5$, in other words, if and only if $|x| \leq 5$.

For a proof by contradiction: assume $|x| > 5$, and suppose for the sake of contradiction that $x^2 \leq 25$. Since $|x| > 5$, we have $x^2 = |x|^2 > 25 = 5^2$, contradicting the supposition that $x^2 \leq 25$.

(d) The contrapositive is: if $x \leq 1$ or $x \geq 5$ then $|x - 3| \geq 2$. Indeed, assume $x \leq 1$ or $x \geq 5$. If $x \leq 1$ then $x - 3 \leq -2$ so that $|x - 3| \geq 2$. If $x \geq 5$, then $x - 3 \geq 2$ so that $|x - 3| \geq 2$. Either way, $|x - 3| \geq 2$.

For a proof by contradiction: assume $|x - 3| < 2$, and suppose for the sake of contradiction that $\neg(1 < x < 5)$. Now $|x - 3| < 2$ means $-2 < x - 3 < 2$, so $1 < x < 5$, a contradiction.

(e) Assume $x^2 = 2$ and for the sake of contradiction, suppose x is rational, so it is possible to write $x = \frac{p}{q}$ in lowest terms (in other words, p and q have no common factors), then $p^2 = 2q^2$, so p^2 is even. Therefore, p is even so $p = 2k$ for some integer k. Then $2q^2 = 4k^2$, showing that q^2 is even, so that q is also even. Then p, q are both even, contradicting the assumption that p and q have no common factors. This contradiction shows that x is must be irrational.

Helpful Tip!

The contrapositive of the statement is easy enough to write out: if x is rational, then $x^2 \neq 2$. However, how would one prove such a statement directly? Any straightforward proof of the contrapositive statement also proceeds via contradiction. It's worthwhile to pause and think on this issue.

(Exercise on page 31.)

Solution for Exercise 21 (Prove or Disprove).

(a) The claim is true. The contrapositive form is: if exactly one of m and n is even and the other odd, then m^2+n^2 is odd. Without loss of generality, assume m is even and n is odd (otherwise, rename the variables). So $m = 2k$ for some integer k and $n = 2\ell + 1$ for some integer ℓ. Then

$$m^2 + n^2 = (2k)^2 + (2\ell + 1)^2 = 2(2k^2 + 2\ell^2 + 2\ell) + 1,$$

which is odd. This proves the contrapositive, which proves the original implication.

Helpful Tip!

The proof above is another example where "without loss of generality" is useful. Whenever you consider using this phrase, it is important to explain why generality isn't lost! For example, in this proof the roles of m and n are symmetric: we say "one of them is even and the other is odd" without declaring which is which; this means we can just call the even one n and the odd one m.

Another hint is to revisit the proof above with the roles of m and n interchanged and verify that it still works; this means we haven't used any "special information" about which variable is m and which one is n. In other words, generality wasn't lost.

(b) The claim is true. Note that $n^2 - n = n(n-1)$ is the product of two consecutive integers, one of which must be even. This follows from our assumption that every integer is either even or odd as follows:

- Suppose n is even, so that $n = 2k$ for some $k \in \mathbb{Z}$. Then $n(n-1) = 2k(n-1) = 2(k(n-1))$ is even.
- Suppose n is odd, so that $n = 2\ell + 1$ for some $\ell \in \mathbb{Z}$. Then $n(n-1) = n(2\ell + 1 - 1) = 2(n\ell)$ is even.

Either way, $n(n-1)$ is even.

(c) The claim is true. There are many ways of proving it; let us try to use what we have already proved. The contrapositive form is: if $m^2 + n^2$ is even, then $m+n$ is even.

Assume m^2+n^2 is even. By part (a) it follows that both m, n are even or both are odd. We prove that in each case $m+n$ is even.

- Suppose both m, n are even, so $m = 2k$ and $n = 2\ell$ for some $k, \ell \in \mathbb{Z}$. Then $m+n = 2(k+\ell)$ is even.
- Suppose both m, n are odd, so $m = 2k+1$ and $n = 2\ell+1$ for some $k, \ell \in \mathbb{Z}$. Then $m+n = 2(k+\ell+1)$ is even.

(d) The claim is true. Note that it consists of two claims:

(i) If mn is even, then one of m, n is even.

(ii) If one of m, n is even, then mn is even.

Let us prove each claim in turn:

(i) We prove the contrapositive form: if both m, n are odd, then mn is odd. Indeed, suppose m, n are odd so that $m = 2k+1$ and $m = 2\ell+1$ for some $k, \ell \in \mathbb{Z}$. Then $mn = (2k+1)(2\ell+1) = 2(2k\ell + k + \ell) + 1$ is odd.

(ii) Suppose without loss of generality m is even, so $m = 2k$ for some $k \in \mathbb{Z}$. Then $mn = 2(kn)$ is even.

(e) The claim is true. Suppose n is odd, so $n = 2k+1$ for some $k \in \mathbb{Z}$. Then $n^2 - 1 = (2k+1)^2 - 1 = 4k(k+1)$. Let us define $m := k+1$, so we may rewrite $n^2 - 1 = 4(m-1)m = 4(m^2 - m)$. In part (b) above, we have shown that $m^2 - m$ must be even, so that $m^2 - m = 2\ell$ for some $\ell \in \mathbb{Z}$ and therefore $n^2 - 1 = 4(2\ell) = 8\ell$, so that $8|(n^2 - 1)$.

(f) Since $x, y \geq 0$ are nonnegative real numbers, each has a well-defined nonnegative square root, $\sqrt{x}, \sqrt{y}$. Then, since the square of any real number is nonnegative, $(\sqrt{x} - \sqrt{y})^2 \geq 0$ and so $x - 2\sqrt{x}\sqrt{y} + y \geq 0$. This implies that $x + y \geq 2\sqrt{xy}$.

(Exercise on page 32.)

Solution for Exercise 22 (Proposition vs Predicate)**.**

Recall that a **proposition** is a statement that is either true or false. A **predicate** is a statement with one or more free variables; a variable is **bound** if it is inside the scope of a quantifier and **free** otherwise. Now that we have all the definitions, we can tackle the problem.

(a) The sun is hot.

This statement has no free variables, so it is not a predicate. The statement's truth value can be evaluated; the sun is in fact hot. Therefore it is a proposition and it is true.

(b) Where is Waldo?

This is a question; we cannot evaluate its truth value, and therefore, it is not a proposition. It also does not have any free variables, and so it is not a predicate either.

(c) $x^2 > 4$.

This statement has a free variable, namely the variable x. We know that x is a free variable because it does not have a value assigned to it. Since we have a free variable in our statement, it cannot be a proposition. If the free variable x is bound, we will be able to evaluate a truth value for the statement (for example, if we have $\forall x \in \mathbb{R}, x^2 > 4$, then the statement is false, and if we have $\exists x \in \mathbb{R}, x^2 > 4$, then the statement is true). It follows that the statement is a predicate with free variable x.

(d) $P(y)$, where $P(y) := y < 1$.

This statement has a free variable y, and so it cannot be a proposition. If y is bound, we will be able to assign a truth value to the statement. Therefore, the statement is a predicate with free variable y.

(e) $Q(0)$, where $Q(x) := x > 1$.

This statement does not have a free variable. We can rewrite it as follows, $0 > 1$. Since it has no free variables, it is not a predicate. We can evaluate the truth value of the statement, since 0 is not greater than 1, the statement is false. Therefore, this is a false proposition.

(f) There exists some integer z such that $2z + 1 = 1$.

There is a variable in this statement, namely z. However, z is bound as it is in the scope of the quantifier "there exists". So the statement has no free variables, and is not a predicate. We can also evaluate the truth value of this statement. We do this by solving the equation we have. If there is some integer z such that $2z + 1 = 1$, we must have $2z = 0$, or $z = 0$. Since $z = 0$ is an integer that satisfies the equation $2z + 1 = 1$, it follows that the quantified statement is true. This makes the statement a true proposition.

(g) For every real number r, $r > 1$.

There is a variable in this statement, namely r. However, r is bound as it is in the scope of the quantifier "for all". So the statement has no free variables, and it is not a predicate. We can also evaluate the truth value of this statement. Since not all real numbers are greater than 1 (for example, 0 is a real number and it is not greater than 1), the statement is false. It follows that the statement is a false proposition.

(Exercise on page 33.)

Solution for Exercise 23 (Vocabulary of Quantifiers).

The **Universal Quantifier** $\forall$ means that every element in the domain of discourse satisfies the condition that follows.

For example, if the domain of discourse is the real numbers, the statement

$$\forall x\,(x^2 \geq 0)$$

asserts (correctly) that the square of any real number is nonnegative. In this case the domain of discourse is implicit. We can make the domain explicit by including it as part of the quantification:

$$\forall x \in \mathbb{R}\,(x^2 \geq 0).$$

A simple example of ungrammatical usage is incorrect symbol placement: $\forall \in x\mathbb{R}$.

Another example of incorrect usage is $\forall x P(y)$. In this case, there is no need for the quantification of x because x is not a free variable in the predicate P.

A very *common error* which combines both of these mistakes has to do with the *scope* of quantification: $(x^2 \geq 0)\forall x \in \mathbb{R}$. The first occurrence of x in that statement is free (unbound). This usage is sometimes seen in informal mathematical writing but should be avoided for the sake of correctness and clarity.

The **Existential Quantifier** $\exists$ means that at least one element from the domain of discourse satisfies the condition that follows.

For example, if the domain of discourse is the integers, the statement

$$\exists x\,(x^2 = 4)$$

asserts (correctly) that there exists an integer whose square is 4; note that there is no claim of uniqueness! Indeed, in this example there are precisely two integers satisfying the condition (namely, $x = \pm 2$).

In the previous example, the domain of discourse is implicit. We can make the domain explicit by making it part of the quantification:

$$\exists x \in \mathbb{Z}\,(x^2 = 4).$$

An example of ungrammatical usage is $\exists x = 2 \in \mathbb{Z}$. The issues become more apparent if we correctly introduce parentheses: $\exists x\,(= 2 \in \mathbb{Z})$. It is clear that $= 2$ is not a well-formed statement! Even if we remedy that issue $\exists x\,(x = 2 \in \mathbb{Z})$ the result is still not a well-formed statement: we have $(x = 2)$ as a statement and $2 \in \mathbb{Z}$ as a statement, but we cannot combine them into one statement. Most likely, the intention behind the original statement was something along the lines of

$$\exists x \in \mathbb{Z}\,(x = 2).$$

Pay attention to the *universe of discourse*, and the *scope of quantification*.

(Exercise on page 34.)

Solution for Exercise 24 (Finite Universe of Discourse)**.**

Let the universe of discourse be the set $A := \{1,2,3,4,5\}$. Recall that $\forall x\, P(x)$ means that *every* element of A satisfies $P(x)$; whereas $\exists x\, P(x)$ means that there is *at least one* $x \in A$ that satisfies $P(x)$.

(a) $\forall x\, P(x)$ means that every element of A satisfies $P(x)$. This means that $P(1)$ is true, and $P(2)$ is true, and so on for every element in A. In other words,

$$P(1) \wedge P(2) \wedge P(3) \wedge P(4) \wedge P(5).$$

(b) $\exists x\, P(x)$ means that there is at least one element in A that satisfies $P(x)$. We do not know which element it is in particular, just that at least one of them satisfies $P(x)$. So either $P(1)$ is true or $P(2)$ is true or $P(3)$ is true or $P(4)$ is true or $P(5)$ is true (where or here is inclusive, meaning it is possible for more than one element in A to satisfy $P(x)$). In other words, we have

$$P(1) \vee P(2) \vee P(3) \vee P(4) \vee P(5).$$

(c) $\neg(\exists x\, P(x))$ means that there is no element in A that satisfies $P(x)$. We can rephrase this by saying that each element of A does not satisfy $P(x)$. Equivalently, each element of A satisfies $\neg P(x)$. So we have

$$\neg P(1) \wedge \neg P(2) \wedge \neg P(3) \wedge \neg P(4) \wedge \neg P(5).$$

If we introduce the predicate $Q(x) := \neg P(x)$, we have

$$Q(1) \wedge Q(2) \wedge Q(3) \wedge Q(4) \wedge Q(5)$$

which is exactly $\forall x\, Q(x)$ (by part (a) above). That is, $\neg(\exists x\, P(x))$ is equivalent to $\forall x\,(\neg P(x))$.

(d) $\neg(\forall x\, P(x))$ means that not every element in A satisfies $P(x)$. We can rephrase this by saying that at least one element of A does not satisfy $P(x)$. Equivalently, at least one element of A satisfies $\neg P(x)$. So we have

$$\neg P(1) \vee \neg P(2) \vee \neg P(3) \vee \neg P(4) \vee \neg P(5).$$

If we introduce the predicate $Q(x) := \neg P(x)$, we have

$$Q(1) \vee Q(2) \vee Q(3) \vee Q(4) \vee Q(5)$$

which is exactly $\exists x\, Q(x)$ (by part (b) above). That is, $\neg(\forall x\, P(x))$ is equivalent to $\exists x\,(\neg P(x))$.

(e) $(\forall x\,((x \neq 3) \implies P(x))) \vee (\exists x\,(\neg P(x)))$ can be split into two statements that are joined by a disjunction. We'll tackle each statement separately then join them again at the end. The first statement is $\forall x((x \neq 3) \implies P(x))$. This statement says that for every $x \in A$, if $x \neq 3$, then $P(x)$ is true. So if $x \in \{1,2,4,5\}$, then $P(x)$ is satisfied, and if $x = 3$, we do not know the truth value of $P(3)$. This is equivalent to

$$P(1) \wedge P(2) \wedge P(4) \wedge P(5).$$

The second statement is $\exists x \neg P(x)$, which means that there is at least one element in A that satisfies $\neg P(x)$. So we have

$$(\neg P(1)) \vee (\neg P(2)) \vee (\neg P(3)) \vee (\neg P(4)) \vee (\neg P(5)).$$

Joining the two statements with a disjunction, we get

$$[P(1) \wedge P(2) \wedge P(4) \wedge P(5)] \vee [(\neg P(1)) \vee (\neg P(2)) \vee (\neg P(3)) \vee (\neg P(4)) \vee (\neg P(5))].$$

For the bonus, if $P(x)$ is the predicate $x > 0$, then we have:

(a) $\forall x\, P(x)$ means that every element of $\{1, 2, 3, 4, 5\}$ is positive, which is true.

(b) $\exists x\, P(x)$ means that least one element of $\{1, 2, 3, 4, 5\}$ is positive, which is true.

(c) $\neg(\exists x\, P(x))$ means that there does not exist a positive integer in the set $\{1, 2, 3, 4, 5\}$, which is false.

(d) $\neg(\forall x\, P(x))$ means that not every element of $\{1, 2, 3, 4, 5\}$ is positive, which is false.

(e) $[\forall x\, ((x \neq 3) \implies P(x))] \vee [\exists x\, (\neg P(x))]$ is true because the first disjunt $\forall x\, ((x \neq 3) \implies P(x))$ is true. Recall that we found this to be equivalent to $P(1) \wedge P(2) \wedge P(4) \wedge P(5)$, which states that the integers 1, 2, 4, and 5 are all positive. Since this is true, the whole statement is true (the truth value of the second disjunct does not matter in this case).

(Exercise on page 35.)

Solution for Exercise 25 (Changing the Universe of Discourse).

(a) $\forall x\,(x^2 \geq 0)$. This statement is true for all three universes of discourse under consideration: $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{R}$.

(b) $\forall x\,(x > -1)$. This statement is true if the universe of discourse is $\mathbb{N}$, since every natural number is nonnegative. However, it is false if the universe of discourse is $\mathbb{Z}$ or $\mathbb{R}$ because these sets contain negative numbers, such as -2, that are not greater than -1. Note that the number -2 is a **counterexample** to this statement, which proves that it is not true.

(c) $\exists x\,(x > 0 \wedge x < 1)$. This statement is true if the universe of discourse is $\mathbb{R}$, because the set of real numbers contains elements strictly between 0 and 1. However, the statement is not true if the universe of discourse is $\mathbb{N}$ or $\mathbb{Z}$, since each of these sets contain only whole numbers.

(d) $\exists x\,(x + 1 < 0)$. This statement is true if the universe of discourse is $\mathbb{Z}$ or $\mathbb{R}$; the number -2 satisfies the predicate $x + 1 < 0$ and $-2 \in \mathbb{Z}$ as well as $-2 \in \mathbb{R}$. However, the statement is false if the universe of discourse is $\mathbb{N}$. For the statement to be true, there must be at least one element satisfying $x + 1 < 0$, but every natural number is positive and adding 1 to it still results in a natural number (hence a positive).

(e) $\forall x\,((x \neq 0) \implies x \text{ is not a solution to } x^2 = 2)$. This statement is true if the universe of discourse is $\mathbb{N}$ or $\mathbb{Z}$. Note that the only solutions to $x^2 = 2$ are $\pm\sqrt{2}$ which are not an integer and therefore not in the set of natural numbers or in the set of integers. However, the statement is false if the universe of discourse is $\mathbb{R}$, with $x = \sqrt{2}$ (for example) a counterexample. Indeed, let us focus on the conditional statement: the antecedent is true $\sqrt{2} \neq 0$, but the consequent is false, $\sqrt{2}$ is a solution to $x^2 = 2$; therefore, the conditional statement is false.

(Exercise on page 36.)

Solution for Exercise 26 (Translating Quantified Statements)**.**

The universe of discourse is all students at UofT, and the predicate $K(x, y)$ is defined by "x knows y".

(a) $\forall x\,\forall y\,K(x, y)$. Every student at UofT knows every student at UofT.

(b) $\forall x, y\,(K(x, y) \implies K(y, x))$. If student x knows student y, then student y also knows student x.

> **Helpful Tip!**
> Note that "$\forall x\,\forall y$" is often contracted to "$\forall x, y$". This can only be done with quantifiers of the same type which appear together; for example, there is no way to contract "$\forall x\,\exists y\,\forall z$".

(c) $\forall x\,\exists y\,K(x, y)$. Every student at UofT knows at least one student at UofT. Note that it is possible that $x = y$ in this case, so that everyone at least knows themselves!

(d) $\exists x\,\forall y\,K(x, y)$. There is at least one student at UofT who knows all the students at UofT.

(e) $\exists x\,\exists y\,K(x, y)$. There is at least one student at UofT who knows at least one student at UofT. Again, it is possible that $y = x$.

(f) $\exists x, y\,((x \neq y) \wedge K(x, y))$. There is at least one student at UofT who knows at least one other student at UofT. Here the possibilty of $y = x$ is ruled out!

(Exercise on page 37.)

Solution for Exercise 27 (Evaluating Quantified Statements).

(a) $\forall x \in \mathbb{R}\, \exists y \in \mathbb{R}\, (x + y = 0)$. This statement asserts that for every real number x there exists a real number y such that $x + y = 0$. This statement is true, for we can always choose $y = -x$.

Reversing the order of the quantifiers we obtain the statement $\exists y \in \mathbb{R}\, \forall x \in \mathbb{R}\, (x + y = 0)$. This statement asserts that there is a real number y such that no matter what real number x is chosen we get $x + y = 0$. This statement is false. The only real number so that $x + y = 0$ is $x = -y$. No matter what y is chosen, we are guaranteed that for at least one of $x_0 = 0$ and $x_1 = 1$, we have $x + y \neq 0$. Indeed, if $x_0 + y = 0$ then $x_1 + y = 1$, whereas if $x_1 + y = 0$ then $x_0 + y = -1$.

(b) $\forall x \in \mathbb{N}\, \exists y \in \mathbb{N}\, (x < y)$. This proposition asserts that for every natural number x there exists at least one natural number y greater than x. The proposition is true. No matter what x is chosen, we can choose $y = x + 1$.

Reversing the order of the quantifiers we obtain the statement $\exists y \in \mathbb{N}\, \forall x \in \mathbb{N}\, (x < y)$. This proposition asserts that there is a natural number y which is greater than all natural numbers. The proposition is false. No matter what value of y is chosen, we can find at least one value of x such that $\neg(x < y)$, for example, $x = y + 1$.

(c) $\forall x \in \mathbb{Z}\, \exists y \in \mathbb{Z}\, (x + y = 7)$. This proposition asserts that for every integer x there exists an integer y such that $x + y = 7$. The proposition is true, for we can choose $y = x - 7$ which is an integer if x is an integer.

Reversing the order of the quantifiers we obtain the proposition $\exists y \in \mathbb{Z}\, \forall x \in \mathbb{Z}\, (x + y = 7)$. This proposition asserts that there is at least one integer y such that no matter what integer x is chosen we have $x + y = 7$. This assertion is false, for no matter what y is chosen we can always choose $x = -y$ and obtain $x + y = 0 \neq 7$, so that it is not true that "for every x we have $x + y = 7$."

(d) $\forall x \in \mathbb{R}\, \exists y \in \mathbb{R}\, (y^2 = x)$. This proposition asserts that every real number x has a real square root: a real number y such that $y^2 = x$. This assertion is false. The real number $x = -1$ has no real number y such that $y^2 = x$.

Reversing the order of the quantifiers, $\exists y \in \mathbb{R}\, \forall x \in \mathbb{R}\, (y^2 = x)$. This proposition asserts that there is one real number y which is a square root of every other real number x. This assertion is also false. No matter what real number y is chosen, we can choose $x = -1$ and be guaranteed that $y^2 \neq -1$.

(e) $\forall x \in \mathbb{R}\, \exists y \in \mathbb{R}\, (y = x^2)$. This proposition asserts that every real number x has a square: a real number y such that $y = x^2$. This assertion is true since we can always choose $y = x^2$.

> **Helpful Tip!**
> Again, it's worth emphasizing that y need not be different from x. Indeed, for $x = 0$ we have $y = 0$ as the only real number which satisfies $y = x^2$.

Reversing the order of quantifiers, $\exists y \in \mathbb{R}\, \forall x \in \mathbb{R}\, (y = x^2)$, we obtain a proposition which asserts that there is (at least) one real number y which is the square of every other real number $y = x^2$. This assertion is false, no matter which y is presented as a candidate for the existential quantifier, we can choose $x = y + 1$. Since $y \neq (y + 1)^2$ for every real number (you should verify this!), we see that y is not the square of every real number.

(Exercise on page 38.)

Solution for Exercise 28 (Exchanging Quantifiers).

(a) It is possible that $\forall x\, \exists y\, P(x, y)$ is true but $\exists x\, \forall y\, P(x, y)$ is false. For example, we can take the domain of discourse to be the real numbers $U = \mathbb{R}$ and $P(x, y)$ the predicate $x + y = 0$.

(b) It is possible that $\forall x\, \exists y\, P(x, y)$ is true but $\exists y\, \forall x\, P(x, y)$ is false. The same example from the previous part works here too. Take the domain of discourse to be the real numbers $U = \mathbb{R}$ and $P(x, y)$ the predicate $x + y = 0$.

(c) It is possible that $\exists x\, \forall y\, P(x, y)$ is true but $\forall x\, \exists y\, P(x, y)$ is false. For example, let the universe of discousre be $U = \{1, 2, 3, 4, 5\}$ and $P(x, y)$ the predicate $(x = 1) \vee (x < y)$.

Then $\exists x\, \forall y\, P(x, y)$ is true; namely, if we choose $x = 1$ then no matter what y is chosen, we always have $P(x, y)$.

On the other hand, $\forall x\, \exists y\, P(x, y)$ is false. Indeed, if we choose $x = 5$ then no matter what y is chosen, the predicate $(x = 1) \vee (x < y)$ is false.

(d) It is **not** possible that $\exists y\, \forall x\, P(x, y)$ is true but $\forall x\, \exists y\, P(x, y)$ is false. Indeed, we claim that the former proposition implies the latter; let us prove this!

Suppose $\exists y\, \forall x\, P(x, y)$ so there is some element $v \in U$ (at least one!) such that no matter what other $u \in U$ is chosen we always have $P(u, v)$.

We wish to prove $\forall x\, \exists y\, P(x, y)$. Towards that end, suppose $x \in U$ is arbitrary. Then we know that $P(x, v)$ is true. Therefore $\exists y\, P(x, y)$ (namely, $y = v$). Since $x \in U$ was arbitrary, we conclude that $\forall x\, \exists y\, P(x, y)$.

Helpful Tip!

This exercise illustrates, among other things, the idea of a *symmetric* predicate: it makes a difference whether $P(x, y) \iff P(y, x)$. We will discuss symmetric relations at length in Chapter 7.

(Exercise on page 39.)

Solution for Exercise 29 (Set-Builder Notation)**.**

(a) We get the elements of our set from $\mathbb{N}$ and the condition they must satisfy is that they are strictly less than 5. Therefore, $\{n \in \mathbb{N} \mid n < 5\} = \{1, 2, 3, 4\}$.

(b) We get the elements of our set from $\mathbb{Z}$ and the condition they must satisfy is that they are between -2 and 2 and do not include -2, but include 2. Therefore, $\{x \in \mathbb{Z} \mid -2 < x \leq 2\} = \{-1, 0, 1, 2\}$.

(c) We can see that all the elements of the set are integers. And they are exactly the even integers, so that is the condition they must satisfy. We can write $\{\dots, -4, -2, 0, 2, 4, \dots\} = \{2n \mid n \in \mathbb{Z}\}$.

(d) First note that the elements of our set are all real numbers, and the condition they must satisfy is that they are striclty greater than 2 and less than or equal to 5. So we have $(2, 5] = \{x \in \mathbb{R} \mid 2 < x \leq 5\}$.

(Exercise on page 41.)

Solution for Exercise 30 (Subsets).

(a) For $A_1 = \{1, 2, 3\}, B_1 = \{1, 2, 3, 4\}$, we have $A_1 \subseteq B_1$ is true, since every element of A_1 is also an element of B_1.

For $A_2 = \{1, 3, 5\}, B_2 = \{2, 4, 6\}$, we have $A_2 \subseteq B_2$ is false since $1 \in A_2$ but $1 \notin B_2$.

For $A_3 = \{\{1\}\}, B_3 = \{1, \{1\}\}$, we have $A_3 \subseteq B_3$ is true because $\{1\} \in \{1, \{1\}\}$. (Note that $1 \neq \{1\}$, so A_3 would not be a subset of $\{1\}$. Make sure not to confuse elements with subsets.)

(b) For example, $A = \{1, 2\}, B = \{1, 2, 3\}$ satisfy $A \subsetneq B$. Notice that all the elements of A are in B, therefore $A \subseteq B$, but $3 \in B$ and $3 \notin A$, so $A \neq B$.

(c) The statement $\emptyset \subseteq A$ is true because there is no element in $\emptyset$ that is not in A. In fact, $\emptyset$ contains no elements at all.

(Exercise on page 42.)

Solution for Exercise 31 (Set Equality).

In order to prove that these two sets are equal, we must show that they are both subsets of each other. Let $A = \{1, 2\}$, and let $B = \{x \in \mathbb{R} \mid x^2 - 3x + 2 = 0\}$,.

- First, we prove that $A \subseteq B$. We need to check that every element of A, namely 1 and 2, is also an element of B. We check each element in turn. Note that 1 is a real number and $1^2 - 3 \cdot 1 + 2 = 0$, so 1 satisfies the condition in the set-builder notation of B. Therefore, $1 \in B$. Next, we see that 2 is a real number and $2^2 - 3 \cdot 2 + 2 = 0$, so 2 satisfies the condition in the set-builder notation of B. Therefore, $2 \in B$. Since all elements of A are contained in B, it follows that $A \subseteq B$.

- Next, we prove that $B \subseteq A$. To do this, we must unpack the set-builder notation and list the elements of B. Note that $b \in B$ if and only if b is a real number and is a solution to equation $x^2 - 3x + 2 = 0$. Factoring the left-hand side gives us $(x - 1)(x - 2) = 0$, and therefore, the only solutions to this equation are 1 and 2. We may list the elements of B and write $B = \{1, 2\}$. Since $1 \in A$ and $2 \in A$, and these are all the elements of B, it follows that $B \subseteq A$.

We have shown that $A \subseteq B$ and $B \subseteq A$. By definition of set equality, we conclude that $A = B$.

(Exercise on page 43.)

Solution for Exercise 32 (Set Operations)**.**

(a) $A \cup B = \{1, 2, 3\} \cup \{2, 3, 4\} = \{x \in U \mid x \in \{1, 2, 3\} \text{ or } x \in \{2, 3, 4\}\} = \{1, 2, 3, 4\}$

(b) $A \cap B = \{1, 2, 3\} \cap \{2, 3, 4\}\{x \in U \mid x \in \{1, 2, 3\} \text{ and } x \in \{2, 3, 4\}\} = \{2, 3\}$

(c) $A \setminus B = \{1, 2, 3\} \setminus \{2, 3, 4\} = \{x \in U \mid x \in \{1, 2, 3\} \text{ and } x \notin \{2, 3, 4\}\} = \{1\}$

(d) $B \setminus A = \{2, 3, 4\} \setminus \{1, 2, 3\} = \{x \in U \mid x \in \{2, 3, 4\} \text{ and } x \notin \{1, 2, 3\}\} = \{4\}$

(e) $A^c = U \setminus A = \{1, 2, 3, 4, 5\} \setminus \{1, 2, 3\} = \{x \in U \mid x \notin \{1, 2, 3\}\} = \{4, 5\}$

(f) $(A^c)^c = U \setminus A^c = \{1, 2, 3, 4, 5\} \setminus \{4, 5\} = \{x \in U \mid x \notin \{4, 5\}\} = \{1, 2, 3\} = A$. Note that this is generally true. We will prove it in one of the following exercise.

(g) $B^c = U \setminus B = \{1, 2, 3, 4, 5\} \setminus \{2, 3, 4\} = \{x \in U \mid x \notin \{2, 3, 4\}\} = \{1, 5\}$

(h) $(A \cup B)^c = U \setminus (A \cup B) = \{1, 2, 3, 4, 5\} \setminus \{1, 2, 3, 4\} = \{x \in U \mid x \notin \{1, 2, 3, 4\}\} = \{5\}$

(i) $A^c \cap B^c = \{4, 5\} \cap \{1, 5\} = \{x \in U \mid x \in \{4, 5\} \text{ and } x \in \{1, 5\}\} = \{5\}$

(j) $(A \cap B)^c = U \setminus (A \cap B) = \{1, 2, 3, 4, 5\} \setminus \{2, 3\} = \{x \in U \mid x \notin \{2, 3\}\} = \{1, 4, 5\}$

(k) $A^c \cup B^c = \{4, 5\} \cup \{1, 5\} = \{x \in U \mid x \in \{4, 5\} \text{ or } x \in \{1, 5\}\} = \{1, 4, 5\}$

Note that $(A \cup B)^c = A^c \cap B^c$ and $(A \cap B)^c = A^c \cup B^c$. These equalities are satisfied for any sets A and B in the same universe; these equalities are called De Morgan's laws.

The sets A and B are not disjoint, because their intersection is not empty; it contains elements 2 and 3. However, the sets A and $B \setminus A$, are disjoint because their intersection is empty.

(Exercise on page 44.)

Solution for Exercise 33 (The Empty Set)**.**

(a) We show that $A \cup \emptyset = A$ by showing 'double subset inclusion', in other words, we show that the two sets on either side of the equality sign are subsets of each other. If $x \in A$, then $x \in A \cup \emptyset$ so $A \subseteq A \cup \emptyset$. Conversely, if $x \in A \cup \emptyset$, then either $x \in A$ or $x \in \emptyset$. But $\emptyset$ is empty, it does not contain any elements, therefore, it must be that $x \in A$. Therefore, $A \cup \emptyset \subseteq A$, and bringing both subset inclusions together, we get $A \cup \emptyset \subseteq A$.

(b) To prove that $A \cap \emptyset = \emptyset$, we proceed as in the previous part. Suppose $x \in A \cap \emptyset$. Then $x \in A$ and $x \in \emptyset$. But there are no elements in $\emptyset$, therefore, x must not exist in $A \cap \emptyset$, and $A \cap \emptyset \subseteq \emptyset$. On the other hand, $\emptyset \subseteq A \cap \emptyset$, in fact, $\emptyset$ is a subset of every set. Therefore, we must have that $A \cap \emptyset = \emptyset$.

(c) We want to show that $\emptyset^c = U$. According to the definition of the complement, $\emptyset^c = U \setminus \emptyset = \{x \in U \mid x \notin \emptyset\}$. But, the empty set contains no elements, therefore, for every $x \in U$, x satisfies the condition that it is not an element of $\emptyset$. It follows that $\emptyset^c = \{x \in U\} = U$.

(Exercise on page 45.)

Solution for Exercise 34 (Properties of Set Operations).

(a) Let A, B and C be sets such that $A \subseteq B$ and $B \subseteq C$. Let $x \in A$, then since $A \subseteq B$, it follows that $x \in B$. And since $B \subseteq C$, all the elements in B are also in C, so that $x \in C$. We have shown that if $x \in A$, then $x \in C$, which is exactly the definition of $A \subseteq C$.

(b) Consider the sets $A = \{1, 2, 4\}, B = \{2, 3\}, C = \{1, 2\}$.

(i) Then

$$\begin{aligned} A \cap (B \cup C) &= \{1, 2, 4\} \cap (\{2, 3\} \cup \{1, 2\}) \\ &= \{1, 2, 4\} \cap \{1, 2, 3\} \\ &= \{1, 2\}. \end{aligned}$$

And

$$\begin{aligned} (A \cap B) \cup (A \cap C) &= (\{1, 2, 4\} \cap \{2, 3\}) \cup (\{1, 2, 4\} \cap \{1, 2\}) \\ &= \{2\} \cup \{1, 2\} \\ &= \{1, 2\} \\ &= A \cap (B \cup C) \end{aligned}$$

(ii) We also have

$$\begin{aligned} A \cup (B \cap C) &= \{1, 2, 4\} \cup (\{2, 3\} \cap \{1, 2\}) \\ &= \{1, 2, 4\} \cup \{2\} \\ &= \{1, 2, 4\}. \end{aligned}$$

And

$$\begin{aligned} (A \cup B) \cap (A \cup C) &= (\{1, 2, 4\} \cup \{2, 3\}) \cap (\{1, 2, 4\} \cup \{1, 2\}) \\ &= \{1, 2, 3, 4\} \cap \{1, 2, 4\} \\ &= \{1, 2, 4\} \\ &= A \cup (B \cap C) \end{aligned}$$

(c) Using the definition of the union operation, we have

$$\begin{aligned} A \cup (B \cup C) &= A \cup \{x \in U \mid x \in B \text{ or } x \in C\} \\ &= \{x \in U \mid x \in A \text{ or } (x \in B \text{ or } x \in C)\} \\ &= \{x \in U \mid x \in A \text{ or } x \in B \text{ or } x \in C\} \\ &= \{x \in U \mid (x \in A \text{ or } x \in B) \text{ or } x \in C\} \\ &= \{x \in U \mid x \in A \text{ or } x \in B\} \cup C \\ &= (A \cup B) \cup C. \end{aligned}$$

(d) Using the definition of the intersection operation, we have

$$\begin{aligned} A \cap (B \cap C) &= A \cap \{x \in U \mid x \in B \text{ and } x \in C\} \\ &= \{x \in U \mid x \in A \text{ and } (x \in B \text{ and } x \in C)\} \\ &= \{x \in U \mid x \in A \text{ and } x \in B \text{ and } x \in C\} \\ &= \{x \in U \mid (x \in A \text{ and } x \in B) \text{ and } x \in C\} \\ &= \{x \in U \mid x \in A \text{ and } x \in B\} \cap C \\ &= (A \cap B) \cap C. \end{aligned}$$

(e) Let $x \in A$, then $x \notin A^c$, and so $x \in (A^c)^c$. Therefore $A \subseteq (A^c)^c$. Conversely, if $x \in (A^c)^c$, then $x \notin A^c$, which implies that $x \in A$. This proves the reverse subset inclusion, $(A^c)^c \subseteq A$. Therefore, $A = (A^c)^c$.

(Exercise on page 46.)

Solution for Exercise 35 (Subset Equivalences)**.**

Each equivalence follows by 'element-chasing'.

First, suppose $A \subseteq B$, so that if $x \in A$, then $x \in B$. In that case,

$$\begin{aligned} A \cup B &= \{x \in U \mid x \in A \text{ or } x \in B\} \\ &= \{x \in U \mid x \in B\} && \text{since } x \in A \implies x \in B \\ &= B. \end{aligned}$$

Conversely, suppose $A \cup B = B$. Let $x \in A$, then $x \in A \cup B = B$, so that $x \in B$. Therefore, $A \subseteq B$.

Next, suppose $A \subseteq B$, so that if $x \in A$, then $x \in B$. In that case,

$$\begin{aligned} A \cap B &= \{x \in U \mid x \in A \text{ and } x \in B\} \\ &= \{x \in U \mid x \in A\} && \text{since } x \in A \implies x \in B \\ &= A. \end{aligned}$$

Conversely, suppose $A \cap B = A$. Let $x \in A$, then $x \in A = A \cap B$, so that $x \in B$. Therefore, $A \subseteq B$.

Bringing this all together, we have,

$$A \subseteq B \iff A \cup B = B \iff A \cap B = A.$$

(Exercise on page 47.)

Solution for Exercise 36 (Set Equalities).

(a) We wish to prove that $U^c = \emptyset$. Recall the definition of the complement; $U^c = U \setminus U = \{x \in U \mid x \notin U\}$. But an element x cannot be in U and not in U simultaneously. So no element satisfies the conditions of this set. Therefore, the set U^c must be empty.

(b) We wish to prove that $A \cap A^c = \emptyset$. Note that

$$\begin{aligned} A \cap A^c &= \{x \in U \mid x \in A \text{ and } x \in A^c\} \\ &= \{x \in U \mid x \in A \text{ and } x \notin A\} && \text{since } A^c = \{x \in U \mid x \notin A\} \\ &= \emptyset && \text{since no element can satisfy both conditions simultaneously.} \end{aligned}$$

(c) We wish to show that $A \cup A^c = U$. Note that

$$\begin{aligned} A \cup A^c &= \{x \in U \mid x \in A \text{ or } x \in A^c\} \\ &= \{x \in U \mid x \in A \text{ or } x \notin A\} && \text{since } A^c = \{x \in U \mid x \notin A\} \\ &= U && \text{since all elements are either in } A \text{ or not in } A. \end{aligned}$$

(d) We are required to prove one of De Morgan's laws, $(A \cup B)^c = A^c \cap B^c$. Suppose $x \in (A \cup B)^c$, then $x \notin A \cup B$. But then $x \notin A$ and $x \notin B$ (otherwise, x would be in their union). In particular, $x \in A^c$ and $x \in B^c$. Therefore, $x \in A^c \cap B^c$, and so $(A \cap B)^c \subseteq A^c \cap B^c$.

Conversely, suppose $x \in A^c \cap B^c$. Then $x \in A^c$ and $x \in B^c$, which means that $x \notin A$ and $x \notin B$. This implies that x is also not in their union, $x \notin A \cup B$. In particular, $x \in (A \cup B)^c$, which shows that $A^c \cap B^c \subseteq (A \cup B)^c$. Therefore, by double subset inclusion, the two sets are equal.

(e) We wish to prove the other De Morgan's law, $(A \cap B)^c = A^c \cup B^c$. We can prove it in the same way we proved the previous law. Or we can use the law we've already proved, combined with the fact that $(A^c)^c = A$, which was proven in Exercise 34, which we will do here. First, replace A and B in the law we proved in the previous part with A^c and B^c, respectively, to get

$$(A^c \cup B^c)^c = (A^c)^c \cap (B^c)^c.$$

Since $(A^c)^c = A$, and $(B^c)^c = B$, we can simplify this to

$$(A^c \cup B^c)^c = A \cap B.$$

Now taking the complement of both sides of this equality, we have

$$((A^c \cup B^c)^c)^c = (A \cap B)^c.$$

Again, since $((A^c \cup B^c)^c)^c = A^c \cup B^c$, we get

$$A^c \cup B^c = (A \cap B)^c.$$

And that is exactly what we wanted to prove.

(Exercise on page 48.)

Solution for Exercise 37 (Union-Complement Form)**.**
Note that

$$\begin{aligned} X \setminus Y &:= \{x \in U \mid x \in X \text{ and } x \notin Y\} \\ &= \{x \in U \mid x \in X \text{ and } x \in Y^c\} \\ &= X \cap Y^c. \end{aligned}$$

(a) We have

$$\begin{aligned} A \setminus (B \cap C) &= A \cap (B \cap C)^c \\ &= A \cap (B^c \cup C^c) && \text{by De Morgan's law} \\ &= (A \cap B^c) \cup (A \cap C^c) && \text{by the Distribution of Union and Intersection} \\ &= ((A \cap B^c)^c)^c \cup ((A \cap C^c)^c)^c && \text{since } (X^c)^c = X \\ &= (A^c \cup (B^c)^c)^c \cup (A^c \cup (C^c)^c)^c && \text{by De Morgan's law} \\ &= (A^c \cup B)^c \cup (A^c \cup C)^c && \text{since } (X^c)^c = X \end{aligned}$$

(b) We have

$$\begin{aligned} (A \setminus B) \cap (C \setminus D) &= (A \cap B^c) \cap (C \cap D^c) \\ &= (A \cap C) \cap (B^c \cap D^c) && \text{since } \cap \text{ is associative and commutative} \\ &= (A \cap C) \cap (B \cup D)^c && \text{by De Morgan's law} \\ &= ((A \cap C)^c)^c \cap (B \cup D)^c && \text{since } (X^c)^c = X \\ &= (A^c \cup C^c)^c \cap (B \cup D)^c && \text{by De Morgan's law} \\ &= (((A^c \cup C^c)^c)^c \cap ((B \cup D)^c)^c)^c && \text{by De Morgan's law} \\ &= ((A^c \cup C^c) \cup (B \cup D))^c && \text{since } (X^c)^c = X. \end{aligned}$$

(Exercise on page 49.)

Solution for Exercise 38 (Symmetric Difference)**.**

(a) Let's break this up. First, we find $A \setminus B := \{x \in U \mid x \in A \text{ and } x \notin B\}$. We have

$$A \setminus B = \{1, 2\}.$$

Next, we find $B \setminus A := \{x \in U \mid x \in B \text{ and } x \notin A\}$. We have

$$B \setminus A = \{4, 5\}.$$

Finally, we find $A \triangle B$, which is the union of the two sets we computed. We have

$$A \triangle B = \{1, 2, 4, 5\}$$

.

(b) Note that $B \triangle A = (B \setminus A) \cup (A \setminus B) = \{4, 5\} \cup \{1, 2\} = \{1, 2, 4, 5\}$

(c) We have

$$\begin{aligned} A \triangle B &= (A \setminus B) \cup (B \setminus A) \\ &= (B \setminus A) \cup (A \setminus B) && \text{since } \cup \text{ is commutative } (X \cup Y = Y \cup X) \\ &= B \triangle A. \end{aligned}$$

(d) Given sets A and B, the set $A \triangle B$ is the set that contains all the elements that are in exactly one of A and B. So it contains the elements of A that are not in B and the elements of B that are not in A.

(e) We have the following

$$\begin{aligned} A \triangle B &= (A \setminus B) \cup (B \setminus A) \\ &= (A \cap B^c) \cup (B \cap A^c) && \text{since } X \setminus Y = X \cap Y^c \\ &= ((A \cap B^c) \cup B) \cap ((A \cap B^c) \cup A^c) && \text{by distributivity} \\ &= ((A \cup B) \cap (B^c \cup B)) \cap ((A \cup A^c) \cap (B^c \cup A^c)) && \text{by distributivity} \\ &= ((A \cup B) \cap U) \cap (U \cap (B^c \cup A^c)) && \text{since } X \cup X^c = U \\ &= (A \cup B) \cap (B^c \cup A^c) && \text{since } X \cap U = X \\ &= (A \cup B) \cap (B \cap A)^c && \text{by de Morgan's law} \\ &= (A \cup B) \setminus (A \cap B) && \text{since } X \setminus Y = X \cap Y^c. \end{aligned}$$

(Exercise on page 50.)

Solution for Exercise 39 (Power Set Definition)**.**

(a) The power set of A is the set of subsets of A.

Note that the empty set, $\emptyset$, is a subset of every set. Next, we write down all subsets of A that contain only 1 element. These are $\{1\}$, and $\{2\}$. Finally, we write down all subsets of A that contain exactly 2 elements. Since A contains exactly 2 elements, $A = \{1, 2\}$ is the only subset of A that contains exactly 2 elements. Putting this together, we get

$$\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}.$$

(b) Let us address each statement in turn.

(i) The statement $\mathcal{P}(A) \subseteq A$ is false. For example, $\{1\} \in \mathcal{P}(A)$ but $\{1\} \notin A$. More generally, the elements of A are integers, while the elements of $\mathcal{P}(A)$ are sets; these are not in the same "universe".

(ii) The statement $\emptyset \subseteq A$ is true because every element of $\emptyset$ is also an element of A. More generally, the same reasoning shows that the empty set is a subset of every set.

(iii) The statement $\emptyset \subseteq \mathcal{P}(A)$ is true. As remarked above, the empty set is a subset of every set.

(iv) The statement $\emptyset \in A$ is false because the only elements of A are 1 and 2. While the empty set is a subset of A, it is not an element of A.

(v) The statement $\emptyset \in \mathcal{P}(A)$ is true. The set $\mathcal{P}(A)$ has four elements and one of these four is $\emptyset$. Recall that $\mathcal{P}(A)$ contains all subsets of A; since the empty set is a subset of A, therefore, it is an element of the power set of A.

(vi) The statement $1 \in A$ is true because 1 and 2 are elements of A.

(vii) The statement $1 \in \mathcal{P}(A)$ is false. The set $\mathcal{P}(A)$ contains four elements and none of them is 1. Recall that $\mathcal{P}(A)$ is the set of subsets of A. Even though 1 is an element of A, it is not a subset of A, so 1 does not belong to the power set of A.

(viii) The statement $\{1\} \in A$ is false because A contains only two elements 1 and 2, and neither of these two elements is $\{1\}$.

(ix) The statement $\{1\} \in \mathcal{P}(A)$ is true. Recall that $\mathcal{P}(A)$ is the set of subsets of A. Since $\{1\}$ is a subset of A (every element in $\{1\}$ is also an element of A), it follows that $\{1\}$ is an element of the power set of A.

(x) The statement $\{1\} \subseteq A$ is true because every element in $\{1\}$ (namely the element 1) is also an element in A.

(xi) The statement $\{1\} \subseteq \mathcal{P}(A)$ is false because there is an element in $\{1\}$ (namely the element 1) that is not an element in $\mathcal{P}(A)$ (the elements in $\mathcal{P}(A)$ are all sets).

(Exercise on page 51.)

Solution for Exercise 40 (Power Set Computation).

Recall that the power set of a set S is the set of subsets of the set S.

(a) $A = \{a\}$. The empty set, $\emptyset$, is a subset of every set. Next, we write down the subsets with exactly 1 element, $\{a\}$. Since A has only 1 element, there are no more subsets, and

$$\mathcal{P}(A) = \{\emptyset, \{a\}\}.$$

(b) $B = \{a, b\}$. The empty set, $\emptyset$, is a subset of every set. Next, we write down the subsets with exactly 1 element, $\{a\}$ and $\{b\}$. Finally, we write down the subsets with exactly 2 elements, $\{a, b\}$. Since B has only 2 elements, there are no more subsets, and

$$\mathcal{P}(B) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}.$$

(c) $C = \{a, \{b\}\}$. The empty set, $\emptyset$, is a subset of every set. Next, we write down the subsets with exactly 1 element, $\{a\}$ and $\{\{b\}\}$. Finally, we write down the subsets with exactly 2 elements, $\{a, \{b\}\}$. Since C has only 2 elements, there are no more subsets, and

$$\mathcal{P}(C) = \{\emptyset, \{a\}, \{\{b\}\}, \{a, \{b\}\}\}.$$

Helpful Tip!

Do not get confused by the extra braces $\{\}$ around the element b. The set C contains exactly two symbols, one of them called a and the other one $\{b\}$. They could have been called $\clubsuit$ and $\diamondsuit$, in which case the power set would have been $\{\emptyset, \{\clubsuit\}, \{\diamondsuit\}, \{\clubsuit, \diamondsuit\}\}$ it makes no difference!

It is also important to note that while $a \in C$ we have $b \notin C$. When we "look inside C" we cannot find the symbol b. Instead, we find the symbol $\{b\}$ so that $\{b\} \in C$. Similarly, $\{b\} \not\subseteq C$ because the set $\{b\}$ has an element which does not appear in C. On the other hand, $\{\{b\}\} \subseteq C$ because every element of $\{\{b\}\}$ (namely, the single element $\{b\}$) appears in C.

(d) $D = \{\emptyset, \{\emptyset\}\}$. The empty set, $\emptyset$, is a subset of every set. Next, we write down the subsets with exactly 1 element, $\{\emptyset\}$ and $\{\{\emptyset\}\}$. Finally, we write down the subsets with exactly 2 elements, $\{\emptyset, \{\emptyset\}\}$. Since D has only 2 elements, there are no more subsets, and

$$\mathcal{P}(D) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}.$$

(e) $E = \mathcal{P}(A)$, where $A = \{a\}$. We found that $\mathcal{P}(A) = \{\emptyset, \{a\}\}$ in the first part of the problem. Now we find the power set of $E = \{\emptyset, \{a\}\}$. The empty set, $\emptyset$, is a subset of every set. Next, we write down the subsets with exactly 1 element, $\{\emptyset\}$ and $\{\{a\}\}$. Finally, we write down the subsets with exactly 2 elements, $\{\emptyset, \{a\}\}$. Since E has only 2 elements, there are no more subsets, and

$$\mathcal{P}(E) = \{\emptyset, \{\emptyset\}, \{\{a\}\}, \{\emptyset, \{a\}\}\}.$$

(f) $F = \{a, b, c\}$. The empty set, $\emptyset$, is a subset of every set. Next, we write down the subsets with exactly 1 element, $\{a\}$, $\{b\}$, and $\{c\}$. Then, we write down the subsets with exactly 2 elements, $\{a, b\}$, $\{a, c\}$, and $\{b, c\}$. Finally, we write down the subsets with exactly 3 elements, $\{a, b, c\}$. Since F has only 3 elements, there are no more subsets, and

$$\mathcal{P}(F) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}.$$

(Exercise on page 52.)

Solution for Exercise 41 (Power Set Cardinality).

Let's look at the power sets we found in Exercise 40 compiled in Table 21.1.

Set	no. elements	Power Set	no. elements
$A = \{a\}$	1	$\{\emptyset, \{a\}\}$	2
$B = \{a, b\}$	2	$\{\emptyset, \{a\}, \{b\}, \{a, b\}\}$	4
$C = \{a, \{b\}\}$	2	$\{\emptyset, \{a\}, \{\{b\}\}, \{a, \{b\}\}\}$	4
$D = \{\emptyset, \{\emptyset\}\}$	2	$\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$	4
$E = \mathcal{P}(A)$	2	$\{\emptyset, \{\emptyset\}, \{\{a\}\}, \{\emptyset, \{a\}\}\}$	4
$F = \{a, b, c\}$	3	$\{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$	8

Table 21.1: Number of Elements in Sets and their Power Sets

We notice that sets with the same number of elements have power sets with the same number of elements; notice how all the sets with 2 elements have power sets with 4 elements.

We also notice that adding one more element to a set doubles the number of elements in its power set. You can see this when looking at sets A, B and F. We can predict that the power set of G will have $2 \times 8 = 16$ elements. (We can check this by finding the power set of G.)

Number of elements in set S	Number of Elements in $\mathcal{P}(S)$
1	2
2	4
3	8
4	16

Table 21.2: Number of Elements in a Set vs its Power Set

From Table 21.2 we can see a pattern. The pattern is that if a set S has n elements, then its power set has 2^n elements. Consequently, we can predict that the power set of $K = \{1, 2, 3, \ldots, k\}$ will have 2^k elements.

Helpful Tip!

We will prove this fact about the size of power sets in Chapter 4. In the meantime, see if you can explain why the size doubles with each new element. When we move from $\{1\}$ to $\{1, 2\}$, what new subsets are added to the power set? What about when we move from $\{1, 2\}$ to $\{1, 2, 3\}$? How about from $\{1, 2, \ldots, k-1\}$ to $\{1, 2, \ldots, k\}$?

(Exercise on page 53.)

Solution for Exercise 42 (Possible Power Sets).

(a) The set $\{1\}$ cannot be a power set. Note that the empty set is a subset of any set and is therefore an element of any power set. Since the empty set is not an element of $\{1\}$, this set cannot be a power set.

(b) The set $\emptyset$ cannot be a power set. As remarked above, any power set contains the empty set and is therefore not empty.

(c) The set $\{\emptyset, \{1\}\}$ is the power set of $\{1\}$, as we saw in the solutions to Exercise 40.

(d) The set $\{\emptyset\}$ is the power set of $\emptyset$. Note that the only subset of $\emptyset$ is the empty set itself!

(e) The set $\{\emptyset, \{1\}, \{\emptyset, 1\}\}$ cannot be a power set.

Suppose for contradiction that $\{\emptyset, \{1\}, \{\emptyset, 1\}\}$ is the power set $\mathcal{P}(S)$ of some set S. Since $\{\emptyset, 1\} \in \mathcal{P}(S)$, then $\{\emptyset, 1\} \subseteq S$. But this means that $\emptyset \in S$, and so $\{\emptyset\} \subseteq S$ and therefore $\{\emptyset\} \in \mathcal{P}(S)$. But $\{\emptyset\} \notin \mathcal{P}(S)$, which contradicts that $\mathcal{P}(S)$ is the power set of S. Therefore, the set $\{\emptyset, \{1\}, \{\emptyset, 1\}\}$ is not a power set.[15]

(f) The set $\{\emptyset, \{1\}, \{2\}\}$ cannot be a power set.

Suppose for contradiction that it is the power set $\mathcal{P}(S)$ of a set S. Since $\{1\} \in \mathcal{P}(S)$, then $\{1\} \subseteq S$, and so $1 \in S$. Similarly, since $\{2\} \in \mathcal{P}(S)$, then $\{2\} \subseteq S$ and $2 \in S$. It follows that $\{1, 2\} \subseteq S$. But $\{1, 2\} \notin \mathcal{P}(S)$, which is a contradiction. Therefore, $\{\emptyset, \{1\}, \{2\}\}$ is not a power set.

(g) $\{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$. This set is the power set of $\{1, 2\}$ as we have found in Exercise 39 above.

(Exercise on page 54.)

[15] Our conclusions from the previous exercise also imply that $\{\emptyset, \{1\}, \{\emptyset, 1\}\}$ is not a power set, since any finite power set must have cardinality which is a power of 2. However, since we haven't proved this fact yet, we are not using it as part of our proofs.

Solution for Exercise 43 (Power Sets Closures)**.**

All parts are proved similarly; let us address each one in turn.

(a) If $X, Y \in \mathcal{P}(S)$ then $X \subseteq S$ and $Y \subseteq S$. We want to prove that $X \cup Y \in \mathcal{P}(S)$, so we need to show that $X \cup Y \subseteq \mathcal{P}(S)$. Towards that end, suppose $s \in X \cup Y$, so that $s \in X$ or $s \in Y$.

- If $s \in X$ then, since $X \subseteq S$, we know that $s \in S$.
- If $s \in Y$ then, since $Y \subseteq S$, we know that $s \in S$.

Either way, $s \in S$ from which we conclude $X \cup Y \subseteq S$.

(b) If $X, Y \in \mathcal{P}(S)$ then $X, Y \subseteq S$. We want to prove that $X \cap Y \in \mathcal{P}(S)$, so we need to show that $X \cap Y \subseteq \mathcal{P}(S)$. Towards that end, suppose $s \in X \cap Y$. Then $s \in X$ and $s \in Y$. Since $s \in X$ and $X \subseteq Y$ we know that $s \in S$. This proves that $X \cap Y \subseteq \mathcal{P}(S)$.

(c) If $X \in \mathcal{P}(S)$, then $X \subseteq S$. We want to prove that $X^c \in \mathcal{P}(S)$, so we need to show that $S \setminus X \subseteq S$. Towards that end, let $s \in S \setminus X$. Then $s \in S$ and $s \notin X$. In particular, $s \in S$, proving that $S \setminus X \subseteq S$.

(d) If $X \in \mathcal{P}(S)$, then $X \subseteq S$. Suppose now $Y \subseteq X$ is some subset of X. We want to prove that $Y \in \mathcal{P}(S)$, so we need to show that $Y \subseteq S$. This follows at once from the transitivity of $\subseteq$, but let us prove it in detail. Let $y \in Y$ be an arbitrary element. Since $Y \subseteq X$, every element of Y is also an element of X, so we must have $y \in X$. Since $X \subseteq S$, every element of X is also an element of S, so we must have $y \in S$. Since $y \in Y$ was arbitrary, this proves that $Y \subseteq S$.

(e) Note that $\{\{1\}\}$ cannot be a power set because it is not downward closed. Indeed, $\{1\} \in \{\{1\}\}$ and $\emptyset \subseteq \{1\}$, but $\emptyset \notin \{\{1\}\}$.

Even though it is downward closed (and even closed under intersections), the set $\{\emptyset, \{1\}, \{2\}\}$ also cannot be a power set since it is not closed under unions. Note that $\{1\}, \{2\} \in \{\emptyset, \{1\}, \{2\}\}$, but $\{1\} \cup \{2\} = \{1, 2\} \notin \{\emptyset, \{1\}, \{2\}\}$.

(Exercise on page 55.)

Solution for Exercise 44 (Set Operations and Power Sets).

(a) $\emptyset \in \mathcal{P}(A)$. The statement is true. The empty set is a subset of every set and is therefore an element of every power set.

(b) If $X \in \mathcal{P}(A)$, then $X \in A$. The statement is false. For a counterexample, let $A = \{1\}$; then $\mathcal{P}(A) = \{\emptyset, \{1\}\}$. Now $\emptyset \in \mathcal{P}(A)$, but $\emptyset \notin A$.

(c) If $A \subseteq B$, then $\mathcal{P}(A) \subseteq \mathcal{P}(B)$. The statement is true. Suppose $X \in \mathcal{P}(A)$, then $X \subseteq A$. But $A \subseteq B$, therefore, $X \subseteq B$ as well[16]. Hence, $X \in \mathcal{P}(B)$, and $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

(d) $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$. The statement is true and we prove it by showing that each set is a subset of the other.

Note that $A \cap B \subseteq A, B$, so by the previous part we have $\mathcal{P}(A \cap B) \subseteq \mathcal{P}(A), \mathcal{P}(B)$ and therefore $\mathcal{P}(A \cap B) \subseteq \mathcal{P}(A) \cap \mathcal{P}(B)$. (You should make sure you understand this last step and can formally prove it using the definitions!)

For the other inclusion $\mathcal{P}(A) \cap \mathcal{P}(B) \subseteq \mathcal{P}(A \cap B)$, let $X \in \mathcal{P}(A) \cap \mathcal{P}(B)$ be some arbitrary element. Then $X \in \mathcal{P}(A)$ and $X \in \mathcal{P}(B)$. Therefore, $X \subseteq A$ and $X \subseteq B$. We claim that $X \subseteq A \cap B$. Indeed, let $x \in X$ be an arbitrary element. Since $X \subseteq A$, we must have $x \in A$. Since $X \subseteq B$, we must have $x \in B$. Therefore, $x \in A$ and $x \in B$ so that $x \in A \cap B$. Since x was an arbitrary element, this proves that $X \subseteq A \cap B$. Since X was an arbitrary set, this proves that $\mathcal{P}(A) \cap \mathcal{P}(B) \subseteq \mathcal{P}(A \cap B)$.

Therefore, $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$.

(e) $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$. The statement is true.

Since $A \subseteq A \cup B$, we have $\mathcal{P}(A) \subseteq \mathcal{P}(A \cup B)$. Similarly, since $B \subseteq A \cup B$, we have $\mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$. We can therefore conclude $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$. (Make sure you understand the last step; it is a good exercise to try and formally prove it from the definitions!)

Alternatively, we can also prove the claim directly: Suppose $X \in \mathcal{P}(A) \cup \mathcal{P}(B)$, then $X \in \mathcal{P}(A)$ or $X \in \mathcal{P}(B)$. Therefore, $X \subseteq A$ or $X \subseteq B$. In the former case $X \subseteq A \subseteq A \cup B$ shows that $X \subseteq A \cup B$. In the latter case, $X \subseteq B \subseteq A \cup B$ shows that $X \subseteq A \cup B$. Either way, $X \subseteq A \cup B$. It follows that $X \in \mathcal{P}(A \cup B)$, and so $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$.

(f) $\mathcal{P}(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(B)$. The statement is false. For a counterexample, consider the sets $A = \{1\}$, and $B = \{2\}$. We have $\mathcal{P}(A) = \{\emptyset, \{1\}\}$, and $\mathcal{P}(B) = \{\emptyset, \{2\}\}$. Now, $A \cup B = \{1, 2\}$, and $\mathcal{P}(A \cup B) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$. But

$$\mathcal{P}(A) \cup \mathcal{P}(B) = \{\emptyset, \{1\}, \{2\}\} \neq \{\emptyset, \{1\}, \{2\}, \{1, 2\}\} = \mathcal{P}(A \cup B).$$

(g) $\mathcal{P}(A^c) = (\mathcal{P}(A))^c$. The statement is false. For a counterexample, consider the set $A = \{1\}$ in the universe $U = \{1, 2, 3\}$. Then $A^c = \{2, 3\}$, and $\mathcal{P}(A^c) = \{\emptyset, \{2\}, \{3\}, \{2, 3\}\}$. Now $\mathcal{P}(U) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$ is the universe we're considering for $\mathcal{P}(A)^c$. Since $\mathcal{P}(A) = \{\emptyset, \{1\}\}$, we have

$$\mathcal{P}(A)^c = \{\{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\} \neq \{\emptyset, \{2\}, \{3\}, \{2, 3\}\} = \mathcal{P}(A^c).$$

[16] This is the transitivity of inclusion. If you are unsure of how to prove this, revisit the answer to Exercise 43(d).

(h) $\mathcal{P}(A) \cap \mathcal{P}(A^c) = \{\emptyset\}$. The statement is true.

We've shown in part (d) above that $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$. Substituting A^c for B, we get

$$\{\emptyset\} = \mathcal{P}(\emptyset) = \mathcal{P}(A \cap A^c) = \mathcal{P}(A) \cap \mathcal{P}(A^c).$$

(Exercise on page 56.)

Solution for Exercise 45 (Finite Indices)**.**

(a) We have
$$\bigcup_{i=1}^{3} S_i = S_1 \cup S_2 \cup S_3 = \{1, 2, 3, 4\}.$$

(b) We have
$$\bigcap_{i=1}^{2} S_i = S_1 \cap S_2 = \{2\}.$$

(c) We have
$$\bigcap_{i=1}^{3} S_i = S_1 \cap S_2 \cap S_3 = \emptyset.$$

(Exercise on page 57.)

Solution for Exercise 46 (Infinite Indices).

(a) Since the value of $\frac{n-1}{n}$ for $n = 1, 2, 3$ is $0, \frac{1}{2}, \frac{2}{3}$ (respectively), we conclude that

$$S_1 = [0, 0] = \{0\}; \qquad S_2 = \left[0, \frac{1}{2}\right]; \qquad S_3 = \left[0, \frac{2}{3}\right].$$

(b) We prove

$$\forall m, n \in \mathbb{N}\,((m < n) \implies (S_m \subseteq S_n)).$$

We use direct proof. Suppose m, n are natural numbers such that $m < n$. For the purpose of proving $S_m \subseteq S_n$, let $x \in S_m$ be an arbitrary element. Then $0 \leq x \leq \frac{m-1}{m}$ by the definition of S_m. Since $m < n$ we have $-n < -m$ and since $nm = mn$, combining these we get $nm - n < mn - m$. That is to say

$$n(m-1) < m(n-1).$$

Since m, n are positive we may divide by them without changing the direction of the inequality:

$$\frac{m-1}{m} < \frac{n-1}{n}.$$

Since $x \leq \frac{m-1}{m}$ we conclude that $x \leq \frac{n-1}{n}$. Therefore, $0 \leq x \leq \frac{n-1}{n}$, so that $x \in S_n$ (by the definition of S_n). Since x was an arbitrary element of S_m, this proves that $S_m \subseteq S_n$.

(c) Intuitively, we have shown in part (b) above that the sets $S_1, S_2, S_3, \ldots$ keep "expanding"; in particular, S_1 is a subset of each of them. We claim

$$\bigcap_{i=1}^{\infty} S_i = S_1 = \{0\}.$$

To prove our claim we need to show that $\bigcap_{i=1}^{\infty} S_i \subseteq \{0\}$ and $\{0\} \subseteq \bigcap_{i=1}^{\infty} S_i$.

- We prove that $\bigcap_{i=1}^{\infty} S_i \subseteq \{0\}$. Towards that end, let $x \in \bigcap_{i=1}^{\infty} S_i$ be an arbitrary element. Then, by the definition of the "big intersection" symbol, $x \in S_i$ for every $i \in \mathbb{N}$. In particular, $x \in S_1 = \{0\}$. Since x was an arbitrary element of $\bigcap_{i=1}^{\infty} S_i$, this proves that $\bigcap_{i=1}^{\infty} S_i \subseteq \{0\}$.
- We prove that $\{0\} \subseteq \bigcap_{i=1}^{\infty} S_i$. For any $i \in \mathbb{N}$ we have $S_i = \left[0, \frac{i-1}{i}\right]$ so that $0 \in S_i$. This proves that $0 \in S_i$ for all $i \in \mathbb{N}$ so that $0 \in \bigcap_{i=1}^{\infty} S_i$. Since 0 is the only element of $\{0\}$, this proves that $\{0\} \subseteq \bigcap_{i=1}^{\infty} S_i$.

(d) Writing $\frac{n-1}{n} = 1 - \frac{1}{n}$ we see that the endpoints of S_n approach 1 as n increases. This suggests that

$$\bigcup_{i=1}^{\infty} S_i = [0, 1).$$

To prove our claim we need to show that $\bigcup_{i=1}^{\infty} S_i \subseteq [0, 1)$ and conversely $[0, 1) \subseteq \bigcup_{i=1}^{\infty} S_i$.

- We prove $\bigcup_{i=1}^{\infty} S_i \subseteq [0, 1)$. Towards that end, let $x \in \bigcup_{i=1}^{\infty} S_i$ be an arbitrary element. By the definition of the "big union" symbol, this means that there exists some $i \in \mathbb{N}$ such that $x \in S_i$. Now, $S_i = \left[0, \frac{i-1}{i}\right)$ and $\frac{i-1}{i} < 1$ (since $i \in \mathbb{N}$ is positive). Therefore, $0 \leq x \leq \frac{i-1}{i} < 1$, which proves that $x \in [0, 1)$. Since x was an arbitrary element of $\bigcup_{i=1}^{\infty} S_i$, this proves that $\bigcup_{i=1}^{\infty} S_i \subseteq [0, 1)$.

- We prove $[0,1) \subseteq \bigcup_{i=1}^{\infty} S_i$. Towards that end, let $x \in [0,1)$ be arbitrary. Since $x < 1$, we have $\varepsilon := 1 - x > 0$. Let $N \in \mathbb{N}$ such that $1/N < \varepsilon$, so that $1/N < 1 - x$ and therefore, $x < 1 - 1/N$. Then $0 \leq x \leq \frac{N-1}{N}$ so that $x \in S_N$. Therefore, there exists $i \in \mathbb{N}$ such that $x \in S_i$ (namely, $i = N$) and we conclude that $x \in \bigcup_{i=1}^{\infty} S_i$. Since x was an arbitrary element of $[0,1)$, this proves that $[0,1) \subseteq \bigcup_{i=1}^{\infty} S_i$.

(Exercise on page 58.)

Solution for Exercise 47 (Uncountable Unions)**.**

Let us tackle each part in turn. If you get stuck on any of the parts and end up reading the answer, we recommend refraining from reading the answers to the parts following. Instead, go back to the problem and try the other parts on your own.

(i) $\bigcup_{r\in\mathbb{R}}\{-r\} = \mathbb{R}$.

(ii) $\bigcup_{r\in\mathbb{R}}\{r^2\} = [0,\infty)$.

(iii) $\bigcup_{r\in\mathbb{R}}\{e^r\} = (0,\infty)$.

(iv) $\bigcup_{r\in\mathbb{R}}\{1\} = \{1\}$.

(v) $\bigcup_{r\in\mathbb{R}}\{r, r^2, -r\} = \mathbb{R}$.

(vi) $\bigcup_{r\in\mathbb{R}}\{1, r\} = \mathbb{R}$.

(vii) $\bigcup_{r\in\mathbb{R}}[0,|r|) = [0,\infty)$.

(viii) $\bigcup_{r\in\mathbb{R}}[-|r|,|r|] = \mathbb{R}$.

(ix) $\bigcup_{r\in\mathbb{R}}(-|r|,|r|) = \mathbb{R}$.

(x) $\bigcup_{r\in\mathbb{R}}\{rm : m\in\mathbb{N}\} = \mathbb{R}$.

(Exercise on page 59.)

Solution for Exercise 48 (Uncountable Unions Revisited).

(a) Let $x \in \bigcup_{r\in\mathbb{R}}\{r^2\}$ be arbitrary. By the definition of the "big union", there exists some $r \in \mathbb{R}$ such that $x \in \{r^2\}$, which is to say $x = r^2$. Since $r^2 \geq 0$, we conclude that $x \in [0, \infty)$.

(b) Let $x \in [0, \infty)$. Since x is a non-negative real number, it has a non-negative square root $\sqrt{x}$. Then $x \in \{x\} = \{(\sqrt{x})^2\}$. Therefore, there exists some $r \in \mathbb{R}$ such that $x \in \{r^2\}$ (namely, $r = \sqrt{x}$). Thus, by the definition of the "big union", $x \in \bigcup_{r\in\mathbb{R}}\{r^2\}$.

(c) We claim that

$$\bigcup_{r\in\mathbb{R}} T_r = \mathbb{R}.$$

In order to prove our assertion we need to show that each of the sets in the above equality is a subset of the other.

- Let $x \in \bigcup_{r\in\mathbb{R}} T_r$ be an arbitrary element. By the definition of the "big union" symbol, there exists some $r \in \mathbb{R}$ such that $x \in T_r = \{r^3\}$, which is to say $x = r^3$. Since $r \in \mathbb{R}$ we also have $r^3 \in \mathbb{R}$, so that $x \in \mathbb{R}$.
- Let $x \in \mathbb{R}$ be an arbitrary element. Every real number has a unique real cubic root, so that $\sqrt[3]{x} \in \mathbb{R}$. Therefore, $x \in \{x\} = \{(\sqrt[3]{x})^3\}$. This proves that there exists some $r \in \mathbb{R}$ such that $x \in T_r$ (namely, $r = \sqrt[3]{x}$). Therefore, by the definition of the "big union" symbol, $x \in \bigcup_{r\in\mathbb{R}} T_r$.

(Exercise on page 60.)

Solution for Exercise 49 (Uncountable Intersection)**.**

Let us tackle each part in turn. If you get stuck on any of the parts and end up reading the answer, we recommend refraining from reading the answers to the parts following. Instead, go back to the problem and try the other parts on your own.

(i) $\bigcap_{r\in\mathbb{R}}\{r\} = \emptyset$.

(ii) $\bigcap_{r\in\mathbb{R}}[-|r|, |r|] = \{0\}$.

(iii) $\bigcap_{r\in\mathbb{R}}(-|r|, |r|) = \emptyset$.

(iv) $\bigcap_{r\in\mathbb{R}}[0, |r|] = \{0\}$.

(v) $\bigcap_{r\in\mathbb{R}}(0, |r|) = \emptyset$.

(vi) $\bigcap_{r\in\mathbb{R}}(-1-|r|, 1+|r|) = (-1, 1)$.

(vii) $\bigcap_{r\in\mathbb{R}}\{m + r : m \in \mathbb{Z}\} = \emptyset$.

(Exercise on page 61.)

Solution for Exercise 50 (Uncountable Intersection Revisited)**.**

(a) Let $r \in \mathbb{R}$ be arbitrary. Then $|r| \geq 0$ so that $-|r| \leq 0 \leq |r|$, which proves $0 \in [-|r|, |r|]$. This shows that $0 \in [-|r|, |r|]$ for any $r \in \mathbb{R}$, so that $0 \in \bigcap_{r\in\mathbb{R}} [-|r|, |r|]$.

(b) Let $x \in \bigcap_{r\in\mathbb{R}} [-|r|, |r|]$ be arbitrary. Then $x \in [-|r|, |r|]$ for every $r \in \mathbb{R}$. In particular, $x \in [-|0|, |0|] = \{0\}$, so that $x = 0$ and therefore $x \in \{0\}$. This proves that $\bigcap_{r\in\mathbb{R}} S_r \subseteq \{0\}$

(c) We claim that

$$\bigcap_{r\in\mathbb{R}} T_r = (-1, 1)$$

- To prove that $\bigcap_{r\in\mathbb{R}} T_r \subseteq (-1, 1)$, let $x \in \bigcap_{r\in\mathbb{R}} T_r$ be arbitrary. By the definition of "big intersection", this means that $x \in T_r$ for every $r \in \mathbb{R}$. In particular, $x \in T_0 = (-1, 1)$. This proves that $\bigcap_{r\in\mathbb{R}} T_r \subseteq (-1, 1)$.
- To prove that $(-1, 1) \subseteq \bigcap_{r\in\mathbb{R}} T_r$, let $x \in (-1, 1)$ be arbitrary. We note that $x \in (-1, 1)$ means that $x < 1$ and also $x > -1$. For any $r \in \mathbb{R}$ we have $|r| \geq 0$ so that $1 + |r| \geq 1$. Since $x < 1$ we may conclude $x < 1 + |r|$. Similarly, $-|r| \leq 0$ so $-1 - |r| \leq -1$. Since $x > -1$ we may conclude $x > -1 - |r|$.

  We have found that $x > -1 - |r|$ and $x < 1 + |r|$, so that $x \in (-1 - |r|, 1 + |r|) = T_r$. Since $r \in \mathbb{R}$ was arbitrary, we see that $x \in T_r$ for every $r \in \mathbb{R}$. By the definition of "big intersection" this means $x \in \bigcap_{r\in\mathbb{R}} T_r$.

(Exercise on page 62.)

Solution for Exercise 51 (Unions and Intersections I).

(a) The intervals I_1, I_2, I_3, I_4 are computed by plugging in $n = 1, 2, 3, 4$ into the definition of I_n. We obtain

$$I_1 = [-1, 3]; \qquad I_2 = \left[\frac{1}{2}, \frac{5}{2}\right]; \qquad I_3 = \left[-\frac{1}{3}, \frac{7}{3}\right]; \qquad I_4 = \left[\frac{1}{4}, \frac{9}{4}\right].$$

(b) In general,

$$I_{2k} = \left[\frac{1}{2k}, \frac{4k+1}{2k}\right]; \qquad I_{2k+1} = \left[-\frac{1}{2k+1}, \frac{4k+3}{2k+1}\right].$$

(c) Let $x \in \left[\frac{1}{2k}, 2\right]$ be arbitrary. That is, we are given that $x \geq \frac{1}{2k}$ and $x \leq 2$. For any $n \geq k$ we have $2n \geq 2k$ and therefore (since both k, n are positive) $\frac{1}{2k} \geq \frac{1}{2n}$. Since $x \geq \frac{1}{2k}$, we may conclude $x \geq \frac{1}{2n}$. Moreover, since n is positive, we have $2 \leq 2 + \frac{1}{n}$. Since $x \leq 2$, we may conclude that $x \leq 2 + \frac{1}{n}$.

We have found that $\frac{1}{2n} \leq x \leq 2 + \frac{1}{n}$, so that $x \in I_{2n}$. Since n was an arbitrary integer greater than k, we conclude that $x \in \bigcap_{n=k}^{\infty} I_{2n} = J_k$.

(d) Let $x \in J_k$ be arbitrary. By the definition of "big intersection", this means $x \in I_{2n}$ for every $n \geq k$. In particular, $x \in I_{2k} = \left[\frac{1}{2k}, \frac{4k+1}{2k}\right]$.

(Exercise on page 63.)

Solution for Exercise 52 (Unions and Intersections II).

(a) We have already shown that $(x \in J_k) \implies x \in \left[\frac{1}{2k}, \frac{4k+1}{2k}\right]$. It remains to show that $x \leq 2$. Assume for contradiction that $x > 2$, so that $\varepsilon := x - 2 > 0$. Let N be sufficiently large so that $1/N < \varepsilon$. Then, since k is positive, $\frac{1}{2(N+k)} < \frac{1}{N} < \varepsilon$. Since $x \in J_k$, we know that $x \in I_{2n}$ for every $n \geq k$; in particular, $x \in I_{2(N+k)} = \left[\frac{1}{2(N+k)}, 2 + \frac{1}{2(N+k)}\right]$. This means that $x \leq 2 + \frac{1}{2(N+k)}$. Therefore,

$$\varepsilon = x - 2 \leq \frac{1}{2(N+k)}$$

contradicting the fact (the choice of N) that $\frac{1}{2(N+k)} < \varepsilon$. This contradiction proves that $x \leq 2$.

(b) We have $J'_k = [0, 2]$, regardless of the value of k.

- We show that $[0, 2] \subseteq J'_k$. Let $x \in [0, 2]$ be arbitrary. For any natural number n we have $[0, 2] \subseteq \left[-\frac{1}{2n+1}, 2 + \frac{1}{n}\right]$. Therefore, $x \in I_{2n+1}$ for any natural number, and in particular for any $n \geq k$. This proves that $x \in \bigcap_{n=k}^{\infty} I_{2n+1} = J'_k$.
- We show that $J'_k \subseteq [0, 2]$. Let $x \in J'_k$ be arbitrary. Then $x \in I_{2n+1}$ for every $n \geq k$.
  The proof that $(x \in J'_k) \implies (x \leq 2)$ is almost identical to the previous part and will be omitted. To prove that $x \geq 0$, assume for contradiction that $x < 0$ so that $\varepsilon := -x > 0$. Let N be sufficiently large so that $1/N < \varepsilon$. Then, since N is positive, $1/(2(N+k)+1) < 1/N < \varepsilon = -x$. It follows that $x < -1/(2(N+k)+1)$ and therefore $x \notin I_{2(N+k)+1}$, contradicting the assumption that $x \in I_{2n+1}$ for every $n \geq k$. This contradiction shows that $x \geq 0$.

(c) Using our results from the previous parts, we see that

$$E_k = \begin{cases} \left[\frac{1}{k}, 2\right] & \text{if } k \text{ is even;} \\ \left[\frac{1}{k+1}, 2\right] & \text{if } k \text{ is odd.} \end{cases}$$

Note that $E_k = J_k \cap J'_{k+1}$ if k is even, and $E_k = J'_k \cap J_{k+1}$ if k is odd.

(d) We claim $\bigcup_{k=1}^{\infty} E_k = (0, 2]$.

- We prove that $(0, 2] \subseteq \bigcup_{k=1}^{\infty} E_k$. Let $x \in (0, 2]$ be arbitrary; in particular, $x > 0$. Let N be sufficiently large so that $1/N < x$. Since N is positive, $\frac{1}{2N} < \frac{1}{N} < x$. Since we also have $x \leq 2$, we conclude $x \in \left[\frac{1}{2N}, 2\right] = E_{2N}$. In particular, there exists some $k \in \mathbb{N}$ such that $x \in E_k$ (namely, $k = 2N$), so that $x \in \bigcup_{k=1}^{\infty} E_k$.
- We prove that $\bigcup_{k=1}^{\infty} E_k \subseteq (0, 2]$. Note that for any positive integer k we have $\left[\frac{1}{k}, 2\right] \subseteq (0, 2]$ and also $\left[\frac{1}{k+1}, 2\right] \subseteq (0, 2]$. Therefore, $E_k \subseteq (0, 2]$ for any positive integer k. This implies that $\bigcup_{k=1}^{\infty} E_k \subseteq (0, 2]$.
  Indeed, let $x \in \bigcup_{k=1}^{\infty} E_k$ be arbitrary. By the definition of the "big union" symbol, there exists some $k \in \mathbb{N}$ such that $x \in E_k$. Since $E_k \subseteq (0, 2]$, we see that $x \in (0, 2]$.

(Exercise on page 64.)

Solution for Exercise 53 (Unions and Intersections III).

Let us denote $H_k := \bigcup_{n=k}^{\infty} I_n$. We claim

$$H_k = \begin{cases} \left[-\frac{1}{k+1}, 2+\frac{1}{k}\right] & \text{if } k \text{ is even;} \\ \\ \left[-\frac{1}{k}, 2+\frac{1}{k}\right] & \text{if } k \text{ is odd.} \end{cases}$$

We prove this claim for k even, the case of k odd is slightly simpler and is left as an exercise for extra practice.

- We prove that $H_k \subseteq \left[-\frac{1}{k+1}, 2+\frac{1}{k}\right]$. Let $x \in H_k$ be arbitrary, so there exists some $n \geq k$ such that $x \in I_k = \left[\frac{(-1)^n}{n}, 2+\frac{1}{n}\right]$; this means that $x \geq \frac{(-1)^n}{n}$ and $x \leq 2+\frac{1}{n}$.

  Since $n \geq k > 0$ we have $2+\frac{1}{n} \leq 2+\frac{1}{k}$, so that $x \leq 2+\frac{1}{n} \leq 2+\frac{1}{k}$.

  Next we consider the parity of n.

  - If n is even, then $\frac{(-1)^n}{n} = \frac{1}{n} > -\frac{1}{k+1}$, so that $x \geq \frac{(-1)^n}{n} \geq -\frac{1}{k+1}$.
  - If n is odd, we must have $n \geq k+1$, since k is even and $n \geq k$. Therefore, $-\frac{1}{k+1} \leq -\frac{1}{n}$ so that $x \geq \frac{(-1)^n}{n} \geq -\frac{1}{k+1}$.

  We see that regardless of the parity of n, $x \geq -\frac{1}{k+1}$ (proof by cases).

  We conclude that $x \in \left[-\frac{1}{k+1}, 2+\frac{1}{k}\right]$.

- We prove that $\left[-\frac{1}{k+1}, 2+\frac{1}{k}\right] \subseteq H_k$. Let $x \in \left[-\frac{1}{k+1}, 2+\frac{1}{k}\right]$ be arbitrary. We distinguish between two cases:

  - If $x \geq \frac{1}{k}$, then we have $x \in \left[\frac{1}{k}, 2+\frac{1}{k}\right] = I_k$. Therefore, there exists some $n \geq k$ for which $x \in I_n$ and we conclude that $x \in \bigcup_{n=k}^{\infty} I_n$.
  - If $x < \frac{1}{k}$, then we have $x \in \left[-\frac{1}{k+1}, 2+\frac{1}{k+1}\right] = I_{k+1}$. Therefore, there exists some $n \geq k$ for which $x \in I_n$ and we conclude that $x \in \bigcup_{n=k}^{\infty} I_n$.

Next, we prove that

$$\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} I_n = \bigcap_{k=1}^{\infty} H_k = [0,2].$$

- Note that $[0,2] \subseteq H_k$ for any $k \in \mathbb{N}$. This implies that $[0,2] \subseteq \bigcap_{k=1}^{\infty} H_k$.
- Let $x \in \bigcap_{k=1}^{\infty} H_k$ be arbitrary; this means that $x \in H_k$ for every $k \in \mathbb{N}$. We want to show that $x \in [0,2]$.

  - Assume for contradiction that $x < 0$. Then $-x > 0$ and there exists some $N \in \mathbb{N}$ for which $1/N < -x$. Therefore, $\frac{1}{2N+1} < \frac{1}{N} < -x$ from which it follows that $x < -\frac{1}{2N+1}$. In particular, $x \notin H_{2N+1} = \left[-\frac{1}{2N+1}, 2+\frac{1}{2N+1}\right]$. This contradicts the assumption that $x \in H_k$ for every $k \in \mathbb{N}$. This contradiction proves that $x \geq 0$.
  - Assume for contradiction that $x > 2$, so that $x-2 > 0$. Then there exists some $N \in \mathbb{N}$ such that $1/N < x-2$ and therefore $x > 2+1/N$. This means that $x \notin H_N$ (regardless of the parity of N), which contradicts the assumption that $x \in H_k$ for every $k \in \mathbb{N}$. This contradiction proves that $x \leq 2$.

(Exercise on page 65.)

Solution for Exercise 54 (Monotone Sequences).

(a) We claim that $\bigcap_{i=1}^{\infty} S_i = S_1$.

- We show that $S_1 \subseteq \bigcap_{i=1}^{\infty} S_i$. Let $x \in S_1$ be arbitrary. We want to show that $x \in S_k$ for every $k \in \mathbb{N}$. Let $k \in \mathbb{N}$ be arbitrary. Then $k \geq 1$ and, since the sequence is increasing, $S_1 \subseteq S_k$. In particular, $x \in S_k$, as we wanted to show.
- We show that $\bigcap_{i=1}^{\infty} S_i \subseteq S_1$. Let $x \in \bigcap_{i=1}^{\infty} S_i$ be arbitrary. Then $x \in S_k$ for every $k \in \mathbb{N}$. In particular, $x \in S_1$.

(b) Since the first set of a decreasing sequence already contains all the other sets, one would guess that $\bigcup_{i=1}^{\infty} S_i = S_1$.

(c) Suppose $\{S_n\}_{n=1}^{\infty}$ is an increasing sequence. We prove that $\{S_n^c\}_{n=1}^{\infty}$ is a decreasing sequence. Towards this end, we need to show that $\forall m, n \in \mathbb{N}.((m < n) \implies (S_n^c \subseteq S_m^c))$.

Let $m, n \in \mathbb{N}$ be arbitrary positive integers and suppose $m < n$. Since $\{S_n\}_{n=1}^{\infty}$ is increasing, we know that $S_n \subseteq S_m$. Therefore, $S_m^c \subseteq S_n^c$.

In a completely analogous fashion one proves that if $\{S_n\}_{n=1}^{\infty}$ is a decreasing sequence, then $\{S_n^c\}_{n=1}^{\infty}$ is an increasing sequence.

(d) Suppose $\{S_n\}_{n=1}^{\infty}$ is a decreasing sequence. Then $\{S_n^c\}_{n=1}^{\infty}$ is an increasing sequence. Therefore, $\bigcap_{n=1}^{\infty} S_n^c = S_1^c$. We conclude that

$$\bigcup_{n=1}^{\infty} S_n = \left(\left(\bigcup_{n=1}^{\infty} S_n\right)^c\right)^c = \left(\bigcap_{n=1}^{\infty} S_n^c\right)^c = (S_1^c)^c = S_1.$$

(Exercise on page 66.)

Solution for Exercise 55 (Pairwise Disjoint)**.**

We prove that if $n \neq \ell$ then $S_n \cap S_\ell = \emptyset$. Suppose $n \neq \ell$ and assume without loss of generality that $\ell < n$ (otherwise, rename the variables). Next, assume for contradiction that $x \in S_n \cap S_\ell$.

- Since $x \in S_n = \left\{\frac{1}{n} + m \,:\, m \in \mathbb{Z}\right\}$, there exists some $m \in \mathbb{Z}$ such that $x = m + \frac{1}{n}$.
- Since $x \in S_\ell = \left\{\frac{1}{\ell} + m \,:\, m \in \mathbb{Z}\right\}$, there exists some $k \in \mathbb{Z}$ such that $x = k + \frac{1}{\ell}$.

We conclude that

$$m + \frac{1}{n} = k + \frac{1}{\ell}. \qquad (\star)$$

We distinguish between three (exhaustive) possibilities:

- If $m < k$, we rearrange Equality $(\star)$ above

  $$k - m + \frac{1}{\ell} - \frac{1}{n} = 0$$

  which is a contradiction, since $(k - m) > 0$ and $\frac{1}{\ell} - \frac{1}{n} > 0$.
- If $m = k$, we obtain from Equality $(\star)$ above $\frac{1}{n} = \frac{1}{\ell}$, which is a contradiction since $\frac{1}{\ell} > \frac{1}{n}$.
- Finally, suppose $m > k$. Then $m - k > 0$ and since $m - k$ is an integer, we have $m - k \geq 1$ (the smallest positive integer). Therefore, $m \geq k+1$. Now, ℓ is a positive integer, so $\ell \geq 1$ and $1/\ell \leq 1$. Therefore, $k + 1 \geq k + \frac{1}{\ell}$ and we conclude that $m \geq k + \frac{1}{\ell}$. Finally, since n is positive, so is $1/n$ and therefore $m + \frac{1}{n} > m$. We conclude that $m + \frac{1}{n} > k + \frac{1}{\ell}$, contradicting Equality $(\star)$ above.

Since for every two real numbers m, k exactly one of $m < k$, $m = k$, or $m > k$ holds, and since each of these leads to a contradiction, we conclude that there is no $x \in S_n \cap S_\ell$. In other words, $S_n \cap S_\ell = \emptyset$.

(Exercise on page 67.)

Solution for Exercise 56 (Limits).

(a) Let $x \in \liminf S_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} S_n$. Then there exists some $k \in \mathbb{N}$ such that $x \in \bigcap_{n=k}^{\infty} S_n$. Therefore, $x \in S_n$ for all $n \geq k$; in other words, $x \in S_j$ for all but finitely many $j \in \mathbb{N}$ (namely, the exceptions are at worst $j = 1, 2, \ldots, k-1$).

Formally, if we take $B = k$ and rename j to n, we get exactly the condition from the question:

$$\exists k \in \mathbb{N}.\forall n \in \mathbb{N}.[(n \geq k) \implies (x \in S_n)].$$

Conversely, suppose $x \in S_j$ for all but finitely many $j \in \mathbb{N}$. That is, let $B \in \mathbb{N}$ be such that $\forall j \in \mathbb{N}.[(j \geq B) \implies (x \in S_j)]$. Then $x \in \bigcap_{j=B}^{\infty} S_j$. Since there exists some $k \in \mathbb{N}$ such that $x \in \bigcap_{j=k}^{\infty} S_n$ (namely, $k = B$), we conclude that $x \in \bigcup_{k=1}^{\infty} \bigcap_{j=k}^{\infty} S_j$.

(b) We proceed in two steps.

- Let $x \in \limsup S_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} S_n$. Then for every $k \in \mathbb{N}$ we have $x \in \bigcup_{n=k}^{\infty} S_n$. That is, for every $k \in \mathbb{N}$ there exists some $n \geq k$ such that $x \in S_n$.
  We wish to prove
  $$\forall B \in \mathbb{N}.\exists j \in \mathbb{N}.[(j \geq B) \wedge (x \in S_j)].$$
  Let $B \in \mathbb{N}$ be arbitrary. We know $x \in \bigcup_{n=B}^{\infty} S_n$. Therefore, there exists some $j \geq B$ such that $x \in S_j$.
- Conversely, suppose $x \in S_j$ for infinitely many $j \in \mathbb{N}$. That is,
  $$\forall B \in \mathbb{N}.\exists j \in \mathbb{N}.[(j \geq B) \wedge (x \in S_j)].$$
  Given any $k \in \mathbb{N}$ there exists some $j \geq k$ such that $x \in S_j$ (this instantiates the universal quantifier above by taking $B = k$). Therefore, $x \in \bigcup_{j=k}^{\infty} S_j$. Since this holds for every $k \in \mathbb{N}$, we conclude that $x \in \bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} S_j$.

(c) Let us rewrite the sets $I_n := \left[\frac{(-1)^n}{n}, 2 + \frac{1}{n}\right]$ as

$$I_n = \begin{cases} \left[-\frac{1}{n}, 2 + \frac{1}{n}\right] & \text{if } n \text{ is odd;} \\ \left[\frac{1}{n}, 2 + \frac{1}{n}\right] & \text{if } n \text{ is even.} \end{cases}$$

It is clear that $0 \in I_n$ whenever n is odd. Therefore, $0 \in I_n$ for infinitely many values of n, and therefore $0 \in \limsup I_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} I_n$.

On the other hand, $0 \notin I_n$ whenever n is even. Therefore, there is no value B such that $0 \in I_n$ for all $n \geq B$. In other words, $0 \notin \liminf I_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} I_n$.

Indeed, if we compare our answers to those of Exercises 51–53, we see that $\limsup I_n = [0, 2]$ whereas $\liminf I_n = (0, 2]$. Try to spend some time "making sense" of these results in light of the new definitions.

(d) Let $x \in \liminf S_n$. Then $x \in S_j$ for all but finitely many $j \in \mathbb{N}$. In particular, $x \in S_j$ for infinitely many $j \in \mathbb{N}$. Therefore, $x \in \limsup S_n$.

Formalizing this reasoning, suppose

$$\exists k \in \mathbb{N}.\forall n \in \mathbb{N}.[(n \geq k) \implies (x \in S_n)].$$

Let $K \in \mathbb{N}$ be such that $\forall n \in \mathbb{N}.[(n \geq K) \implies (x \in S_n)]$. We wish to prove

$$\forall B \in \mathbb{N}.\exists j \in \mathbb{N}.[(j \geq B) \wedge (x \in S_j)].$$

Let $B \in \mathbb{N}$ be arbitrary. Since $B + K \geq K$, we must have $x \in S_{B+K}$. Therefore, we have found some j such that $j \geq B$ and $x \in S_j$ (namely, $j = B + K$).

(e) If $\{S_n\}_{n\in\mathbb{N}}$ is a pairwise disjoint collection, we have $\limsup S_n = \liminf S_n = \emptyset$.

To prove that $\liminf S_n = \emptyset$, assume for contradiction that $x \in \liminf S_n$. That is,

$$\exists k \in \mathbb{N}.\forall n \in \mathbb{N}.[(n \geq k) \implies (x \in S_n)].$$

Let $K \in \mathbb{N}$ be such that $\forall n \in \mathbb{N}.[(n \geq K) \implies (x \in S_n)]$. Since $K, K+1 \geq K$ we must have $x \in S_K$ and $x \in S_{K+1}$. Then $x \in S_K \cap S_{K+1}$. But $\{S_n\}_{n\in\mathbb{N}}$ is a pairwise disjoint collection, so $S_K \cap S_{K+1} = \emptyset$ (since $K \neq K+1$), a contradiction.

Similarly, to prove that $\limsup S_n = \emptyset$, assume for contradiction that $x \in \limsup S_n$. That is,

$$\forall B \in \mathbb{N}.\exists j \in \mathbb{N}.[(j \geq B) \wedge (x \in S_j)].$$

Instantiate $B = 1$ and let j_1 be such that $j_1 \geq 1$ and $x \in S_{j_1}$. Instantiate $B = j_1 + 1$ and let j_2 be such that $j_2 \geq j_1 + 1$ and $x \in S_{j_2}$. Then $x \in S_{j_1} \cap S_{j_2}$. But $\{S_n\}_{n\in\mathbb{N}}$ is a pairwise disjoint collection, so $S_{j_1} \cap S_{j_2} = \emptyset$ (since $j_1 \neq j_2$), a contradiction.

(f) Suppose $\{S_n\}_{n\in\mathbb{N}}$ is an increasing sequence. If x is an element of all but finitely many of the S_n, then it is clearly an element of $\bigcup_{n=1}^{\infty} S_n$, and similarly if x is an element of infinitely many of the sets in the sequence. We therefore claim that

$$\limsup S_n = \liminf S_n = \bigcup_{n=1}^{\infty} S_n. \qquad (\star\star)$$

There are many approaches to proving this claim! We are looking for one that would make use of everything we've already show. We know that

$$\liminf S_n \subseteq \limsup S_n.$$

Therefore, if we prove that $\limsup S_n \subseteq \bigcup_{n=1}^{\infty} S_n$ and that $\bigcup_{n=1}^{\infty} S_n \subseteq \liminf S_n$, we would prove Equality $(\star\star)$; make sure you understand why this is the case!

- Let us prove that $\limsup S_n \subseteq \bigcup_{n=1}^{\infty} S_n$. Suppose that $x \in \limsup S_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} S_n$. Then for every $k \in \mathbb{N}$ we know that $x \in \bigcup_{n=k}^{\infty} S_n$. In particular, for $k = 1$ we have that $x \in \bigcup_{n=1}^{\infty} S_n$.[17]
- Let us prove that $\bigcup_{n=1}^{\infty} S_n \subseteq \liminf S_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} S_n$. Suppose $x \in \bigcup_{n=1}^{\infty} S_n$. Then there exists some $k \in \mathbb{N}$ such that $x \in S_k$. Since $\{S_n\}_{n=1}^{\infty}$ is increasing, we know that

  $$\forall j \in \mathbb{N}.[(j \geq k) \implies (x \in S_j)].$$

  In words, $x \in S_j$ for every $j \geq k$; this exactly means that $x \in \bigcap_{n=k}^{\infty} S_n$. Therefore, there exists some $k \in \mathbb{N}$ such that $x \in \bigcap_{n=k}^{\infty} S_n$, so that $x \in \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} S_n$.

[17] Note that so far we haven't used the assumption that $\{S_n\}_{n=1}^{\infty}$ is increasing. Since Equality $(\star\star)$ relies on this fact, this is a strong hint that we'd have to make use of it soon!

(g) Suppose $\{S_n\}_{n\in\mathbb{N}}$ is a decreasing sequence. We can write a very similar proof to that in the previous part to show that

$$\limsup S_n = \liminf S_n = \bigcap_{n=1}^{\infty} S_n.$$

This is good practice and you are encouraged to do so! Since the proof looks similar, there should also be some sort of shortcut that would allow us to use what we have already proved, and this is indeed the case. Just as in Exercise 54, we can use the Generalized DeMorgan Laws!

We know from Exercise 54 that if $\{S_n\}_{n\in\mathbb{N}}$ is decreasing, then $\{S_n^c\}_{n\in\mathbb{N}}$ is increasing, so by the previous part

$$\limsup S_n^c = \liminf S_n^c = \bigcup_{n=1}^{\infty} S_n^c.$$

Therefore,

$$(\limsup S_n^c)^c = (\liminf S_n^c)^c = \left(\bigcup_{n=1}^{\infty} S_n^c\right)^c.$$

Now,

$$(\limsup S_n^c)^c = \left(\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} S_n^c\right)^c = \bigcup_{k=1}^{\infty}\left(\bigcup_{n=k}^{\infty} S_n^c\right)^c = \bigcup_{k=1}^{\infty}\bigcap_{n=k}^{\infty}(S_n^c)^c = \bigcup_{k=1}^{\infty}\bigcap_{n=k}^{\infty} S_n = \liminf S_n.$$

Similarly,

$$(\liminf S_n^c)^c = \limsup S_n.$$

Finally,

$$\left(\bigcup_{n=1}^{\infty} S_n^c\right)^c = \bigcap_{n=1}^{\infty} S_n.$$

Putting everything together we have

$$\liminf S_n = \limsup S_n = \bigcap_{n=1}^{\infty} S_n$$

as we wanted to prove.

(h) Suppose $\{S_n\}_{n\in\mathbb{N}}$ is a sequence such that $S_1 = S_3 = S_5 = \cdots$ and also $S_2 = S_4 = S_6 = \cdots$. Let us call S_o (for "odd") the common value of $S_1, S_3, S_5, \ldots$ and S_e (for "even") the common value of $S_2, S_4, S_6, \ldots$.

Intuitively, if x is an element of all but finitely many of the sets, then x is an element of some odd-indexed set and also of some even-indexed set, so that $x \in S_o \cap S_e$. This suggests

$$\liminf S_n = S_o \cap S_e.$$

The argument above is already an outline of the proof, but let us formalize it.

- Suppose $x \in \liminf S_n = \bigcup_{k=1}^{\infty}\bigcap_{n=k}^{\infty} S_n$. Therefore, there exists some $k \in \mathbb{N}$ such that $x \in \bigcap_{n=k}^{\infty} S_n$. That is, $x \in S_n$ for all $n \geq k$. In particular, $x \in S_{2k} = S_e$ and also $x \in S_{2k+1} = S_o$. Therefore, $x \in S_o \cap S_e$.

- Suppose $x \in S_o \cap S_e$. We claim that $x \in \bigcap_{n=1}^{\infty} S_n$; that is, that $x \in S_n$ for every $n \in \mathbb{N}$. Indeed, if n is odd, then $S_n = S_o$ and so $x \in S_n$. If n is even, then $S_n = S_e$, so $x \in S_n$. Since every natural number is either odd or even, this exhausts all options so that $x \in S_n$ for every $n \in \mathbb{N}$. Therefore, we have found some $k \in \mathbb{N}$ such that $x \in \bigcap_{n=k}^{\infty} S_n$ (namely, $k = 1$) so that $x \in \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} S_n$.

Similarly, if x is an element of infinitely many of the sets, then either x is in some odd-indexed set and therefore in S_o, or x is in some even-indexed set and therefore in S_e, or both! This suggests

$$\limsup S_n = S_o \cup S_e.$$

We can write a very similar proof to the one above, or we can use DeMorgan Laws again! Indeed, $S_o^c = S_1^c = S_3^c = S_5^c = \cdots$ and also $S_e^c = S_2^c = S_4^c = S_6^c = \cdots$. Therefore, by what we just proven,

$$\liminf S_n^c = S_o^c \cap S_e^c.$$

It follows that

$$\limsup S_n = (\liminf S_n^c)^c = (S_o^c \cap S_e^c)^c = S_o \cup S_e.$$

(Exercise on page 68.)

Solution for Exercise 57 (Tuples vs. Sets)**.**

Sets are defined solely in terms of their elements, so if two sets have precisely the same elements, they are identical. For example, to prove that $\{a, b\} = \{b, a\}$ we prove that every element on the left is also an element on the right (i.e., that $\{a, b\} \subseteq \{b, a\}$) and vice versa (i.e., that $\{b, a\} \subseteq \{a, b\}$).

Tuples are defined in terms of their elements and the order of their elements. That is, $(u, v) = (x, y)$ if and only if $u = x$ and $v = y$. In particular, if $a \neq b$ then $(a, b) \neq (b, a)$.

(Exercise on page 69.)

Solution for Exercise 58 (Computing Products)**.**

(a) For $A = \{0,1\}$ and $B = \{-1,1\}$, we have

   (i) $(0,-1)$ is an element of $A \times B$, because $0 \in A$ and $-1 \in B$.
   (ii) $(1,2)$ is not an element of $A \times B$ because $2 \notin B$.
   (iii) $(-1,1)$ is not an element of $A \times B$ because $-1 \notin A$.
   (iv) $(1,1)$ is an element of $A \times B$ because $1 \in A$ and $1 \in B$.
   (v) $(0,0)$ is not an element of $A \times B$ because $0 \notin B$.

   For completeness, we can list all the elements of $A \times B$,

$$A \times B = \{(0,-1);(0,1);(1,-1);(1,1)\}.$$

(b) Let $A = \{1,2\}$ and $B = \{3,4,5\}$.

   (i) We have
$$A \times B = \{(1,3);(1,4);(1,5);(2,3);(2,4);(2,5)\}.$$
   (ii) We have
$$B \times A = \{(3,1);(3,2);(4,1);(4,2);(5,1);(5,2)\}.$$
   (iii) Note that $A \times B \neq B \times A$. Indeed, in this example, neither is a subset of the other. For instance, $(1,3) \in A \times B$ and $(1,3) \notin A \times B$; similarly, $(3,1) \in B \times A$ and $(3,1) \notin A \times B$.

(c) Let $A = \{x,y,z\}$ and $B = \{1\}$. Then,

$$A \times B = \{(x,1);(y,1);(z,1)\}.$$

(d) Let A be an arbitrary set and $B = \{b\}$. Then,

$$A \times B = \{(a,b) \,:\, a \in A\}.$$

(Exercise on page 70.)

Solution for Exercise 59 (Empty Products).

Let us address each part in turn.

(a) Suppose $A = \emptyset$ and assume for contradiction that $A \times B \neq \emptyset$. Let $(a, b) \in A \times B = \{(a, b) : a \in A, b \in B\}$ be an arbitrary element. Then $a \in A$ by the definition of $A \times B$; but $A = \emptyset$, so this is a contradiction.

(b) Suppose $B = \emptyset$ and assume for contradiction that $A \times B \neq \emptyset$. Let $(a, b) \in A \times B = \{(a, b) : a \in A, b \in B\}$ be an arbitrary element. Then $b \in B$ by the definition of $A \times B$; but $B = \emptyset$, so this is a contradiction.

(c) We prove the contrapositive. Suppose $A \neq \emptyset$ and $B \neq \emptyset$. Then there exists at least one element in A, say $a \in A$. Similarly, there exists at least one element in B, say $b \in B$. It follows that $(a, b) \in \{(a, b) : a \in A, b \in B\} = A \times B$. That is, there exists at least one element in $A \times B$ so that $A \times B \neq \emptyset$.

(Exercise on page 71.)

Solution for Exercise 60 (Properties of Cartesian Products).

Let us address each part in turn.

(a) **Commutativity.** An example of sets A, B for which $A \times B \neq B \times A$ is $A = \{1\}$ and $B = \{2\}$ (almost any two sets will do). Indeed,

$$A \times B = \{(1, 2)\} \neq \{(2, 1)\} = B \times A.$$

(b) An example of sets A, B for which $A \times B = B \times A$ is when $A = B$. Another example is when one of A, B is the empty set. We shall prove in the next exercises that these are the only examples.

(c) **Associativity.** Suppose $A, B, C \neq \emptyset$ so that each of these sets has at least one element, say $a \in A$, $b \in B$, and $c \in C$. To prove that $(A \times B) \times C \neq A \times (B \times C)$ we note that $((a, b), c) \in (A \times B) \times C$, but $((a, b), c) \notin A \times (B \times C)$.

Indeed, $((a, b), c) \in (A \times B) \times C$ because $(a, b) \in A \times B$ (because $a \in A$ and $b \in B$) and $c \in C$. In contrast, $((a, b), c) \notin A \times (B \times C)$ because $(a, b) \notin A$.

One could also show that $(a, (b, c)) \in A \times (B \times C)$ but $(a, (b, c)) \notin (A \times B) \times C$. Therefore, neither of these sets $(A \times B) \times C$ and $A \times (B \times C)$ is a subset of the other[18].

We note that if one of the sets A, B, C is empty, then both products are empty and are therefore equal. For example, if $A = \emptyset$ then $(A \times B) = \emptyset$ (see Exercise 59) and therefore $(A \times B) \times C = \emptyset$. Moreover, since $A = \emptyset$, we have $A \times (B \times C) = \emptyset$.

(d) **Cancellation.** Suppose $A \neq \emptyset$ and $A \times B = A \times C$. Let us prove that $B = C$.

First note that if $B = \emptyset$, then $A \times B = \emptyset = A \times C$. By Exercise 59, it follows that either A or C is empty. Since we assumed A is non-empty, we must have $C = \emptyset = B$.

Next, we consider the case when B is non-empty. To prove that $B \subseteq C$, let $b \in B$ be an arbitrary element. Since $A \neq \emptyset$, there is at least one element $a \in A$. It follows that $(a, b) \in A \times B$. Since $A \times B = A \times C$, we know that $(a, b) \in A \times C$ so that $a \in A$ and $b \in C$. In particular, $b \in C$. Since $b \in B$ was arbitrary, we conclude that $B \subseteq C$.

The proof that $C \subseteq B$ is completely symmetric (interchange the roles of B and C in the proof above).

For an example where the conclusion fails when $A = \emptyset$ we may choose $B = \{2\}$ and $C = \{3\}$. Then $A \times B = \emptyset = A \times C$ but $B \neq C$.

(Exercise on page 72.)

[18]Nevertheless, there is a natural way to identify these two sets via a bijection $((a, b), c) \leftrightarrow (a, (b, c))$. We shall learn more about bijections in Chapter 7.

Solution for Exercise 61 (Criteria for Commutativity).

Let us address each part in turn.

(a) Suppose that $A \subseteq C$ and $B \subseteq D$. To prove that $A \times B \subseteq C \times D$, let $(a, b) \in A \times B$ be an arbitrary element. Then $a \in A \subseteq C$ and $b \in B \subseteq D$ so that $a \in C$ and $b \in D$; therefore, $(a, b) \in C \times D$. Since $(a, b) \in A \times B$ was an arbitrary element, this proves that $A \times B \subseteq C \times D$.

(b) Suppose $A, B \neq \emptyset$ and $A \times B \subseteq C \times D$. We wish to prove that $(A \subseteq C) \wedge (B \subseteq D)$. Let us prove that $A \subseteq C$, the proof that $B \subseteq D$ is completely analogous.

Towards proving $A \subseteq C$ we start with an arbitrary element $a \in A$. Since $B \neq \emptyset$, we know it has at least one element, say $b \in B$. Then $(a, b) \in A \times B \subseteq C \times D$. Therefore, $(a, b) \in C \times D$ so that $a \in C$ and $b \in D$. In particular, $a \in C$. Since $a \in A$ was an arbitrary element, we conclude that $A \subseteq C$, as we wanted to show.

Note that if $B = \emptyset$ the proof above does not work. Moreover, the claim fails: for example, if $A = \{1, 2\}$, $B = \emptyset$, $C = \{1\}$ and $D = \{2\}$ we have

$$A \times B = \emptyset \subseteq \{(1, 2)\} = C \times D$$

but $A \not\subseteq C$. Similar counterexamples can be constructed if $A = \emptyset$. (However, if both $A, B = \emptyset$ then we of course have $A \subseteq C$ and $B \subseteq D$ vacuously!)

(c) In one direction, it is clear that if (at least) one of A, B is empty, then $A \times B = \emptyset = B \times A$. Moreover, if $A = B$ then we also clearly have $A \times B = A \times A = B \times A$.

For the converse, suppose $A \times B = B \times A$ and $A, B \neq \emptyset$; we need to prove that $A = B$. If we define $C := B$ and $D := A$, then from $A \times B = C \times D$ and the fact that $A, B \neq \emptyset$, we may use the previous part of this exercise to conclude $A \subseteq C$ and $B \subseteq D$. Plugging in our definition of C, D, this just means that $A \subseteq B$ and $B \subseteq A$, i.e., $A = B$.

(Exercise on page 73.)

Solution for Exercise 62 (Product Projections)**.**

Let us address each part in turn.

(a) Similarly to how we solved Exercise 58, we can work out that $A \times B = \{(1,3);(1,4);(2,3);(2,4)\}$ has four elements. A set of four elements has 2^4 subsets, so that is the number of choices for S. The set S may be empty, or it may be the full set $A \times B$ or anything in between. In other words, S can have any size between zero and four.

(b) Let's argue that $\pi_A(\emptyset) = \pi_B(\emptyset) = \emptyset$. Indeed, if you suppose otherwise, say that $a \in \pi_A(\emptyset)$, then there would exist $b \in B$ such that $(a,b) \in \emptyset$, which is clearly absurd. The argument for $\pi_B(\emptyset)$ is analogous. If S is a singleton, say $S = \{(1,3)\}$, then $\pi_A(S) = \{1\}$ and $\pi_B(S) = \{3\}$; $\pi_A(\{(1,3);(2,3)\}) = \{1,2\}$ and $\pi_B(\{(1,3);(2,3)\}) = \{3\}$; $\pi_A(\{(1,3);(1,4);(2,3)\}) = \{1,2\}$ and $\pi_B(\{(1,3);(1,4);(2,3)\}) = \{3,4\}$; finally, $\pi_A(A \times B) = A$ and $\pi_B(A \times B) = B$.

(c) Let A, B and $S \subseteq A \times B$ be arbitrary. From the definition, we have $S \subseteq \pi_A(S) \times \pi_B(S)$. For the other inclusion, we have the following problem. Given $(a,b) \in \pi_A(S) \times \pi_B(S)$, the definition gives us that $(a,b_0),(a_0,b) \in S$ for some $(a_0,b_0) \in A \times B$, which is not exactly what we need to conclude that $(a,b) \in S$. What could go wrong? It might be that no valid choice of b_0 equals b or that the same happens for a_0. This idea in fact gives us a quick counterexample: Let $A = B = \mathbb{N}$ and $S = \{(0,0),(1,1)\}$, then $\pi_A(S) = \pi_B(S) = \{0,1\}$ so that $\pi_A(S) \times \pi_B(S)$ contains $(0,1)$, and is thus not equal to S.

(d) Continuing from above, a necessary and sufficient condition is when the existence of $(a_0,b_0) \in A \times B$ (as deduced above) implies that $(a,b) \in S$. This happens precisely when S contains all possible pairs of its projected elements. Intuitively, when S is a rectangle. Let's formalize this and prove that equality holds precisely when S is of the form $X \times Y$ for some $X \subseteq A$ and $Y \subseteq B$. Assuming this condition, $\pi_A(S) = X$ and $\pi_B(S) = Y$ and so the equality holds. Conversely, if the equality holds, let $X := \pi_A(S)$ and $Y := \pi_B(S)$. Clearly, $S = X \times Y$.

(Exercise on page 74.)

Solution for Exercise 63 (Visualizing Products)**.**

Let us address each part in turn.

(a) The region is a rectangle containing the top and bottom boundaries, but not the lateral sides.

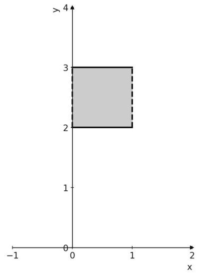


Figure 21.3: $(0, 1) \times [2, 3]$

(b) The region is the top left quarter of the plane.

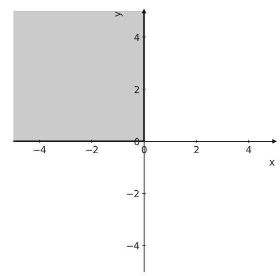


Figure 21.4: $(-\infty, 0] \times [0, \infty)$

(c) The first region ($\mathbb{R} \times \mathbb{N}$) consists of horizontal lines that pass through every natural number on the y-axis. The second region consists of vertical lines that pass through every natural number on the x-axis. They are not equal, but they intersect at every point of $\mathbb{N} \times \mathbb{N}$.

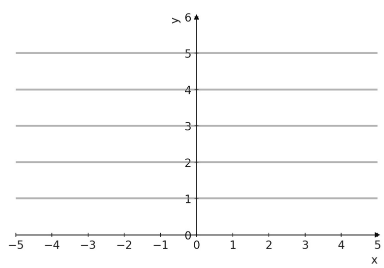


Figure 21.5: $\mathbb{R} \times \mathbb{N}$

(Exercise on page 75.)

Solution for Exercise 64 (Distributivity of the Product I).

Let us address each part in turn.

(a)

$$\begin{aligned} A \times (B \cup C) &= A \times \{2, 3, 4\} \\ &= \{(0, 2); (0, 3); (0, 4); (1, 2); (1, 3); (1, 4)\} \end{aligned}$$

while

$$\begin{aligned} (A \times B) \cup (A \times C) &= \{(0, 2); (0, 3); (1, 2); (1, 3)\} \cup \{(0, 3); (0, 4); (1, 3); (1, 4)\} \\ &= \{(0, 2); (0, 3); (1, 2); (1, 3); (0, 4); (1, 4)\}. \end{aligned}$$

(b)

$$\begin{aligned} A \times (B \cap C) &= A \times \{3\} \\ &= \{(0, 3); (1, 3)\} \end{aligned}$$

while

$$\begin{aligned} (A \times B) \cap (A \times C) &= \{(0, 2); (0, 3); (1, 2); (1, 3)\} \cap \{(0, 3); (0, 4); (1, 3); (1, 4)\} \\ &= \{(0, 3); (1, 3)\}. \end{aligned}$$

(c)

$$\begin{aligned} A \times (B \setminus C) &= A \times \{2\} \\ &= \{(0, 2); (1, 2)\} \end{aligned}$$

while

$$\begin{aligned} (A \times B) \setminus (A \times C) &= \{(0, 2); (0, 3); (1, 2); (1, 3)\} \setminus \{(0, 3); (0, 4); (1, 3); (1, 4)\} \\ &= \{(0, 2); (1, 2)\}. \end{aligned}$$

(Exercise on page 76.)

Solution for Exercise 65 (Distributivity of the Product II).

Let us address each part in turn.

(a) For all x and y, notice that $(x, y) \in A \times (B \cup C)$ if and only if $x \in A$ and $y \in B \cup C$. This is equivalent to the statement $x \in A$ and $y \in B$ or $y \in C$, which is the same as saying that either $x \in A$ and $y \in B$ or $x \in A$ and $y \in C$, equivalently, $(x, y) \in (A \times B) \cup (A \times C)$.

(b) Replace every instance of the word 'or' with 'and', and the symbol $\cup$ with $\cap$ in the above proof.

(c) Notice that for any $(a, b) \in A \times (B \setminus C)$, $a \in A$ and $b \in B \setminus C$. Consequently, $(a, b) \in A \times B$ but $(a, b) \notin A \times C$ since $b \notin C$. Thus, $(a, b) \in (A \times B) \setminus (A \times C)$, and the first inclusion is proved. In particular, we have shown that $A \times (B \setminus C) \subseteq (A \times B) \setminus (A \times C)$. Similarly, any $(a, b) \in (A \times B) \setminus (A \times C)$ must be an element of $A \times B$, so that $a \in A$, and $b \in B$. Now, if b were in C, then $(a, b) \in A \times C$ which is false. Hence, $b \notin C$, and $b \in B \setminus C$, thus $(a, b) \in A \times (B \setminus C)$. Therefore, $(A \times B) \setminus (A \times C) \subseteq A \times (B \setminus C)$. Since we have show the double subset inclusion, the two sets must in fact be equal, and we have $A \times (B \setminus C) = (A \times B) \setminus (A \times C)$ as required.

(d) With the tools developed above,

$$\begin{aligned}
A \times (B \triangle C) &= A \times [(B \cup C) \setminus (B \cap C)] && \text{by definition} \\
&= [A \times (B \cup C)] \setminus [A \times (B \cap C)] && \text{by (c)} \\
&= [(A \times B) \cup (A \times C)] \setminus [(A \times B) \cap (A \times C)] && \text{by (a) and (b)} \\
&= (A \times B) \triangle (A \times C) && \text{by definition}
\end{aligned}$$

(e) Yes. The proofs are analogous to the ones above.

(Exercise on page 77.)

Solution for Exercise 66 (Product and other Set Operations)**.**

Let us address each part in turn.

(a) Given x and y, observe that $(x, y) \in (A \times C) \cap (B \times D)$ if and only if the following four conditions all hold simultaneously: $x \in A$, $x \in C$, $y \in B$, and $y \in D$. Equivalently, $(x, y) \in (A \cap B) \times (C \cap D)$.

(b) Let $A = C = \{0\}$ and $B = D = \emptyset$. Then $A \times C$ is a singleton set, so that $(A \times C) \cup (B \times D)$ is nonempty. But of course $(A \cup B) \times (C \cup D) = A \times \emptyset = \emptyset$.

(c) We first note that $(A \times C) \cup (B \times D) \subseteq (A \cup B) \times (C \cup D)$ always holds. Suppose that (x, y) is an element on the left-hand side, meaning either $(x, y) \in A \times C$ or $(x, y) \in B \times D$. In either case, $x \in A \cup B$ and $y \in C \cup D$. Therefore, (x, y) is an element of the right-hand side.

Part (b) of this exercise shows that the other containment might be false.

Now suppose that $(A \cup B) \times (C \cup D) \subseteq (A \times C) \cup (B \times D)$. Observe that $A \times D,\ B \times C \subseteq (A \cup B) \times (C \cup D) \subseteq (A \times C) \cup (B \times D)$. Therefore, the equality holds if and only if $A \times D,\ B \times C \subseteq (A \times C) \cup (B \times D)$.

If, in addition, $A \cap B = C \cap D = \emptyset$ and $a_0 \in A$, then for every $d \in D$, $(a_0, d) \in A \times D$, which by the previous paragraph implies that $(a_0, d) \in A \times C$ (since $a_0 \notin B$), where $d \in C$. Thus, $D \subseteq C$. Similarly, $C \subseteq D$. Therefore, in this case the equality holds precisely when $C = D$.

(Exercise on page 78.)

Solution for Exercise 67 (Product and other Set Operations II)**.**

Let us address each part in turn.

(a) Let's say the universe is $U = V = \mathbb{R}$. Then $\mathbb{Q} \times \mathbb{Q}$ consists of all points in the plane whose coordinates are rational. Hence, a point is in $(\mathbb{Q} \times \mathbb{Q})^c$ if at least one of its coordinates is irrational. But $\mathbb{Q}^c \times \mathbb{Q}^c$ only contains points where **both** coordinates are irrational. Meaning, $(\sqrt{2}, 0)$ is a member of the former, but not the latter.

(b) For any $(x, y) \in U \times V$, $(x, y) \in X \times Y$ if and only if $x \in X$ and $y \in Y$. Negating this, $(x, y) \in (X \times Y)^c$ if and only if $x \in X^c$ or $y \in Y^c$, in other words, $(x, y) \in (X^c \times V) \cup (U \times Y^c)$.

(Exercise on page 79.)

Solution for Exercise 68 (Distributivity Revisited).

Let us address each part in turn.

(a) For every a and b, $(a,b) \in \left(\bigcup_{i\in I} A_i\right) \times B$ if and only if there exists $i \in I$ with $a \in A_i$, and $b \in B$; but this is clearly equivalent to $(a,b) \in A_i \times B$ for some $i \in I$ or, in other words, $(a,b) \in \bigcup_{i\in I}(A_i \times B)$.

(b) Replace every instance of the expressions 'for some' and 'there exists' with 'for all', and replace the appropriate symbols.

(c) The proofs are analogous.

(Exercise on page 80.)

Solution for Exercise 69 (Inductive reasoning). (a) We are given that $P(1)$ is true and that for every natural number n, $P(n) \implies P(n+1)$. Substituting $n = 1$, we get that $P(1) \implies P(2)$. Since $P(1)$ is true, it follows that $P(2)$ is also true.

Substituting $n = 2$ to the universally-quantified statement, we get that $P(2) \implies P(3)$. We have just shown that $P(2)$ is true, so it follows that $P(3)$ is true.

Similarly, substituting $n = 3$ into $P(n) \implies P(n+1)$, we get that $P(3) \implies P(4)$. Since we've already shown that $P(3)$ is true, this means that $P(4)$ must also be true.

(b) In order to prove $P(100)$ we will continue with the reasoning above. Having proved $P(4)$, we will plug-in $n = 4$ into $P(n) \implies P(n+1)$ to obtain $P(4) \implies P(5)$ and conclude that $P(5)$ is true. We continue plugging in $n = 5$ to prove $P(6)$, then $n = 6$ to prove $P(7)$, and so on until (after a total of 99 "steps") we plug in $n = 99$ to prove $P(100)$.

(c) By the argument above, we can see that each statement $P(n)$ implies the following statement $P(n+1)$. We start with $P(1)$ being true and this implies $P(2)$, which implies $P(3)$, which implies $P(4)$, which implies $P(5)$, which implies $P(6)$, and we keep going on through all the natural numbers to get that $P(n)$ is true for every $n \in \mathbb{N}$. In general, for the natural number m we can prove $P(m)$ in $m - 1$ such "steps" using this technique.

(d) While we can show that for any particular natural number m, the statement $P(m)$ is true, it will take us $m - 1$ steps to do so. If we wanted to prove this for all natural numbers *using this particular reasoning*, it will take an infinite number of steps, but a proof must be completed in a finite amount of steps!

The idea behind adding this axiom is that we can "see" that the pattern of proof continues: we think there can be no counter-examples because for any "challenge" of the form "Prove $P(m)$" we know how to answer it. However, our mathematical proof system doesn't have this "bird's-eye view". Therefore, we add an axiom formalizing our idea that all such challenges can be met.

(This is a rather philosophical question and is worth thinking about. We give very direct and intuitive answer here, perhaps as a starting point for future discussions!)

(Exercise on page 81.)

Solution for Exercise 70 (Recap).

(a) We compute $T_1 = 1$, $T_2 = 3$, $T_3 = 6$, $T_4 = 10$, $T_5 = 15$.

(b) The predicate $P(n)$ is

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

Another way of writing this is

$$\sum_{k=1}^{n} k = T_n.$$

(c) Plugging in 1 into the predicate, we get the assertion

$$1 = \frac{1(1+1)}{2}$$

which is true by computation.

(d) Plugging in $n+1$ into the predicate we get the assertion

$$1 + 2 + \cdots + (n+1) = \frac{(n+1)(n+1+1)}{2}.$$

(e) Using direct proof, means we are *assuming* $P(n)$, i.e., that

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

That is the **inductive hypothesis**. We wish to prove $P(n+1)$, which we wrote in the previous part. Using the inductive hypothesis we have

$$\begin{aligned} 1 + 2 + \cdots + (n+1) &= \frac{n(n+1)}{2} + (n+1) \\ &= \frac{n(n+1) + 2(n+1)}{(n+1)} \\ &= \frac{(n+1)(n+2)}{2}. \end{aligned}$$

This proves $P(n)$.

(f) We have verified $P(1)$ and have shown that $\forall n \in \mathbb{N}(P(n) \implies P(n+1))$. It follows by mathematical induction that $\forall n \in \mathbb{N}\, P(n)$.

The three key steps in any induction proof are

i. Verify the **base case**, $P(1)$.

ii. Assume the **inductive hypothesis**: suppose that $P(n)$ is true for some $n \geq 1$.

iii. Complete the **inductive step**: prove that $P(n+1)$ is true using your assumptions.

(Exercise on page 82.)

Solution for Exercise 71 (Writing inductive proofs).

(a) The first five cases are

$$\begin{aligned} 1 &= 1^2 \\ 1+3 &= 2^2 \\ 1+3+5 &= 3^2 \\ 1+3+5+7 &= 4^2 \\ 1+3+5+7+9 &= 5^2. \end{aligned}$$

(b) The predicate $P(n)$ is the equation

$$1+3+5+\cdots+(2n-1)=n^2.$$

It can also be written as

$$\sum_{k=1}^{n}(2k-1)=n^2.$$

(c) $P(1)$ is the assertion

$$1=1^2$$

which is clearly true.

(d) $P(n+1)$ is the assertion

$$1+3+5+\cdots+(2n+1)=(n+1)^2.$$

(e) We assume $P(n)$ and prove $P(n+1)$. Using the induction hypothesis $P(n)$ we have

$$1+3+5+\cdots+(2n-1)+(2n+1)=n^2+(2n+1)=(n+1)^2.$$

In summary, we have verified $P(1)$ and have shown $\forall n\in\mathbb{N}(P(n)\implies P(n+1))$. By mathematical induction we conclude $\forall n\in\mathbb{N}\,P(n)$. That is, the sum of the first n odd natural numbers is the n-th square number.

(Exercise on page 83.)

Solution for Exercise 72 (Writing inductive proofs II)**.**
We define the predicate $P(n)$ by

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}.$$

Another way of writing this is

$$\sum_{k=1}^{n} k(k+1) = \frac{n(n+1)(n+2)}{3}.$$

We shall prove $\forall n \in \mathbb{N}\, P(n)$ by mathematical induction.
Base Case. The statement $P(1)$ asserts

$$1 \cdot 2 = \frac{1(2)(3)}{3},$$

which is clearly true.
Assume $P(n)$ for some $n \geq 1$, that is assume

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}.$$

(This is the **inductive hypothesis**.) We prove $P(n+1)$. The statement $P(n+1)$ asserts

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) + (n+1)(n+2) = \frac{(n+1)(n+2)(n+3)}{3}.$$

Using the inductive hypothesis,

$$\begin{aligned} 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) + (n+1)(n+2) &= \frac{n(n+1)(n+2)}{3} + (n+1)(n+2) \\ &= \frac{n(n+1)(n+2) + 3(n+1)(n+2)}{3} \\ &= \frac{(n+1)(n+2)(n+3)}{3}. \end{aligned}$$

This proves $P(n+1)$ and (since n was arbitrary) completes the proof that $\forall n \in \mathbb{N}(P(n) \implies P(n+1))$. In conclusion, we have verified $P(1)$ and have shown that $\forall n \in \mathbb{N}(P(n) \implies P(n+1))$. By mathematical induction we conclude that $\forall n \in \mathbb{N}\, P(n)$.

(Exercise on page 84.)

Solution for Exercise 73 (False proof).

(a) $P(n+1)$ asserts the equality

$$1 + 2 + 4 + \cdots + 2^n + 2^{n+1} = 2^{n+2} + 1.$$

(b) Suppose $P(n)$ for some $n \geq 1$, that is suppose

$$1 + 2 + 4 + \cdots + 2^n = 2^{n+1} + 1.$$

Then,

$$1 + 2 + 4 + \cdots + 2^n + 2^{n+1} = 2^{n+1} + 1 + 2^{n+1} = 2 \cdot 2^{n+1} + 1 = 2^{n+2} + 1$$

proving $P(n+1)$. Since n was an arbitrary natural number, we have shown $\forall n \in \mathbb{N}(P(n) \implies P(n+1))$.

(c) $P(3)$ is the assertion

$$1 + 2 + 4 + 8 = 2^4 + 1.$$

That assertion is **false** as the left-side adds up to 15 and the right-side equals 17.

(d) We haven't shown $\forall n\, P(n)$ by induction because we are missing the base case. Indeed, $P(1)$ is false. Even more generally, there is no n for which $P(n)$ is true!

(e) Consider the predicate $Q(n)$, defined by the equation

$$1 + 2 + 4 + \cdots + 2^n = 2^{n+1} - 1.$$

One can prove by mathematical induction that $\forall n \in \mathbb{N}\, Q(n)$; we encourage you do try this for practice. In fact, the inductive step is virtually identical to the one from part (b) above!

(Exercise on page 85.)

Solution for Exercise 74 (Asymptotic growth)**.** Let $P(n)$ be the predicate

$$n! > 2^n.$$

We find the smallest b for which $P(b)$ is true[19]:

$$\begin{aligned} 1! &= 1 < 2 = 2^1 \\ 2! &= 2 < 4 = 2^2 \\ 3! &= 6 < 8 = 2^3 \\ 4! &= 24 > 16 = 2^4. \end{aligned}$$

We now prove by mathematical induction $\forall n \in \mathbb{N}\,(n \geq 4 \implies P(n))$. Note that we have already verified the base case $n = 4$. To prove the inductive step $\forall n \geq 4(P(n) \implies P(n+1))$, suppose that for some $n \geq 4$ we have $P(n)$, that is we are assuming $n! > 2^n$. We wish to prove $P(n+1)$, which is the assertion that $(n+1)! > 2^{n+1}$. Using the inductive hypothesis,

$$\begin{aligned} (n+1)! &= 1 \cdot 2 \cdot 3 \cdots n \cdot (n+1) \\ &= n! \cdot (n+1) \\ &> 2^n \cdot (n+1) \\ &> 2^n \cdot 2 \\ &= 2^{n+1} \end{aligned}$$

which proves $P(n+1)$. (Note that we've used the inductive hypothesis in the third line and the fourth line is justified by the fact that $n \geq 4$ so $n+1 > 2$.)

In summary, we have verified $P(4)$ and $\forall n \in N(n \geq 4 \implies (P(n) \implies P(n+1)))$ so by mathematical induction we conclude $\forall n \in \mathbb{N}(n \geq 4 \implies P(n))$.

Helpful Tip!

More generally, no matter what base $a \in \mathbb{N}$ is chosen, we eventually have $n! > a^n$; we say that the factorial function, $f(n) = n!$, grows *asymptotically faster* than any exponential function, a^n.

(Exercise on page 87.)

[19]Note that the question does not ask for the smallest such b, just for *any* lower bound. In this case, the computation is simple enough so we may as well find the first lower bound, but in other questions the flexibility of not taking the smallest possible value may be useful.

Solution for Exercise 75 (Asymptotic growth II)**.**

Let $P(n)$ be the predicate

$$n^n > n!$$

Note that $P(1)$ is false but $P(2)$ is true, as $2^2 = 4 > 2 = 2!$.

We shall prove by mathematical induction that $\forall n \in \mathbb{N}(n \geq 2 \implies P(n))$. We have already verified the base case $P(2)$, and we now prove the inductive step $\forall n \geq 2(P(n) \implies P(n+1))$.

Towards that end, suppose $P(n)$ holds for some $n \geq 2$. That is, we suppose $n^n > n!$. We wish to prove $P(n+1)$, that is $(n+1)^{n+1} > (n+1)!$.

It is not immediately apparent how we can make use of the inductive hypothesis, so we have to create an opportunity to do so. We note that $(n+1) > n$ so that[20] $(n+1)^m > n^m$. Therefore,

$$\begin{aligned}(n+1)^{n+1} &= (n+1)^n \cdot (n+1) \\ &> n^n \cdot (n+1) \\ &> n! \cdot (n+1) \quad \text{by the inductive hypothesis} \\ &= 1 \cdot 2 \cdot 3 \cdots n \cdot (n+1) \\ &= (n+1)!\end{aligned}$$

which proves $P(n)$. In summary, we have verified $P(2)$ and $\forall n \in N(n \geq 2 \implies (P(n) \implies P(n+1)))$ so by mathematical induction we conclude $\forall n \in \mathbb{N}(n \geq 2 \implies P(n))$.

(Exercise on page 88.)

[20]This assertion should be "obvious" from algebra, but can itself be proved by induction from the rule that if $a > b$ and $c > 0$ then $ac > bc$. Such proofs are important for the axiomatic development of number systems.

Solution for Exercise 76 (Convergence)**.**

(a) We compute

$$\begin{aligned}1!! &= 1\\ 2!! &= 2\\ 3!! &= 3\cdot 1 = 3\\ 4!! &= 4\cdot 2 = 8\\ 5!! &= 5\cdot 3\cdot 1 = 15\\ 6!! &= 6\cdot 4\cdot 2 = 48\\ 7!! &= 7\cdot 5\cdot 3\cdot 1 = 105\\ 8!! &= 8\cdot 6\cdot 4\cdot 2 = 384\\ 9!! &= 9\cdot 7\cdot 5\cdot 3\cdot 1 = 945\\ 10!! &= 10\cdot 8\cdot 6\cdot 4\cdot 2 = 3840.\end{aligned}$$

(b) Using the definition, we see that

$$\begin{aligned}(n+2)!! &= (n+2)\cdot n\cdot(n-2)\cdot(n-4)\cdots a\\ &= (n+2)\cdot n!!\end{aligned}$$

The initial conditions are $1!! = 1$ and $2!! = 2$.

(c) We can use our computations from part (a) above (or, alternatively, the recursion from part (b)):

$$a_1 = \frac{1!!}{2!!} = \frac{1}{2}, \qquad a_2 = \frac{3!!}{4!!} = \frac{3}{8}, \qquad a_3 = \frac{5!!}{6!!} = \frac{5}{16},$$
$$a_4 = \frac{7!!}{8!!} = \frac{35}{128}, \qquad a_5 = \frac{9!!}{10!!} = \frac{63}{256}.$$

(d) Using our recursive relation for the double-factorial from part (b) above,

$$a_{n+1} = \frac{(2n+1)!!}{(2n+2)!!} = \frac{2n+1}{2n+2}\cdot\frac{(2n-1)!!}{(2n)!!} = \frac{2n+1}{2n+2}\cdot a_n.$$

(e) Let $P(n)$ be the predicate[21]

$$\frac{1}{\sqrt{4n}} \le a_n \le \frac{1}{\sqrt{2n+1}}.$$

Base case. We have computed that $a_1 = \frac{1}{2}$ and since $\sqrt{3} < \sqrt{4} = 2$, it is clear that $\frac{1}{2} \le a_1 \le \frac{1}{\sqrt{3}}$.

Inductive step. We prove that $\forall n \in \mathbb{N}\,(P(n) \implies P(n+1))$. Towards that end, let $n \in \mathbb{N}$ be arbitrary and suppose $P(n)$ holds, that is, suppose $\frac{1}{\sqrt{4n}} \le a_n \le \frac{1}{\sqrt{2n+1}}$. We now prove $P(n+1)$, that is, we prove $\frac{1}{\sqrt{4n+4}} \le a_{n+1} \le \frac{1}{\sqrt{2n+3}}$.

Using the inductive hypothesis we have

$$\frac{2n+1}{2n+2}\cdot\frac{1}{\sqrt{4n}} \le a_{n+1} \le \frac{2n+1}{2n+2}\cdot\frac{1}{\sqrt{2n+1}}.$$

[21] Formally, we can write $P(n)$ as

$$\left(\frac{1}{\sqrt{4n}} \le a_n\right) \wedge \left(a_n \le \frac{1}{\sqrt{2n+1}}\right)$$

to emphasize the logical structure of the predicate.

Therefore, suffice it to show that $\frac{1}{\sqrt{4n+4}} \leq \frac{2n+1}{2n+2} \cdot \frac{1}{\sqrt{4n}}$ and that $\frac{2n+1}{2n+2} \cdot \frac{1}{\sqrt{2n+1}} \leq \frac{1}{\sqrt{2n+3}}$ because then we could conclude

$$\frac{1}{\sqrt{4n+4}} \leq \frac{2n+1}{2n+2} \cdot \frac{1}{\sqrt{4n}} \leq a_{n+1} \leq \frac{2n+1}{2n+2} \cdot \frac{1}{\sqrt{2n+1}} \leq \frac{1}{\sqrt{2n+3}}.$$

- We prove that $\frac{1}{\sqrt{4n+4}} \leq \frac{2n+1}{2n+2} \cdot \frac{1}{\sqrt{4n}}$. This is equivalent to proving

  $$(2n+2)\sqrt{4n} \leq (2n+1)\sqrt{4n+4}.$$

  Since both sides of the inequality are positive, we may square both sides and conclude that the inequality holds if and only if

  $$\begin{aligned} &4n(2n+2)^2 \leq (4n+4)(2n+1)^2 \\ &\iff \\ &4n[(2n+2)^2 - (2n+1)^2] \leq 4(2n+1)^2 \\ &\iff \\ &n(4n+3) \leq (2n+1)^2 \\ &\iff \\ &0 \leq n+1 \end{aligned}$$

  which obviously holds.
- We prove that $\frac{2n+1}{2n+2} \cdot \frac{1}{\sqrt{2n+1}} \leq \frac{1}{\sqrt{2n+3}}$. This is equivalent to proving that

  $$(2n+1)\sqrt{2n+3} \leq (2n+2)\sqrt{2n+1}.$$

  Again, because both sides are positive we may square each side and conclude that the inequality above holds if and only if the inequalities below hold.

  $$\begin{aligned} &(2n+3)(2n+1)^2 \leq (2n+1)(2n+2)^2 \\ &\iff \\ &0 \leq (2n+1)[(2n+2)^2 - (2n+1)(2n+3)] \\ &\iff \\ &0 \leq 1 \end{aligned}$$

  which again obviously holds.

This concludes the proof by induction that for every natural number $n \in \mathbb{N}$,

$$\frac{1}{\sqrt{4n}} \leq a_n \leq \frac{1}{\sqrt{2n+1}}.$$

Since $\lim_{n\to\infty} \frac{1}{\sqrt{4n}} = \lim_{n\to\infty} \frac{1}{\sqrt{2n+1}} = 0$ we conclude by the Squeeze Theorem that $\lim_{n\to\infty} a_n = 0$.

Helpful Tip!

Note that $a_{n+1} = a_n \cdot \frac{2n+1}{2n+2} < a_n$, so that $\{a_n\}_{n=1}^{\infty}$ is a decreasing sequence of positive numbers and therefore must converge. The bounds above help us to prove that the limit is in fact 0.

(Exercise on page 89.)

Solution for Exercise 77 (Convergence II).

(a) Computing the first few values of the product we find

$$\prod_{k=1}^{1}\left(1+\frac{(-1)^n}{n+1}\right)=\left(1-\frac{1}{2}\right)=\frac{1}{2}.$$
$$\prod_{k=1}^{2}\left(1+\frac{(-1)^n}{n+1}\right)=\left(1-\frac{1}{2}\right)\left(1+\frac{1}{3}\right)=\frac{2}{3}.$$
$$\prod_{k=1}^{3}\left(1+\frac{(-1)^n}{n+1}\right)=\left(1-\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1-\frac{1}{4}\right)=\frac{1}{2}.$$
$$\prod_{k=1}^{4}\left(1+\frac{(-1)^n}{n+1}\right)=\left(1-\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\left(1+\frac{1}{5}\right)=\frac{3}{5}.$$
$$\prod_{k=1}^{5}\left(1+\frac{(-1)^n}{n+1}\right)=\left(1-\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\left(1+\frac{1}{5}\right)\left(1-\frac{1}{6}\right)=\frac{1}{2}.$$
$$\prod_{k=1}^{6}\left(1+\frac{(-1)^n}{n+1}\right)=\left(1-\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\left(1+\frac{1}{5}\right)\left(1-\frac{1}{6}\right)\left(1+\frac{1}{7}\right)=\frac{4}{7}.$$

(b) We conjecture that $p_{2n+1}=\frac{1}{2}$ and that $p_{2n}=\frac{n+1}{2n+1}$.

(c) Let $P(n)$ be the predicate $p_{2n-1}=\frac{1}{2}$, that is

$$\prod_{k=1}^{2n-1}\left(1+\frac{(-1)^n}{n+1}\right)=\frac{1}{2}.$$

We have already verified the base case $P(1)$ since $p_1=\frac{1}{2}$. To prove the inductive step, we assume that for some n we have $p_{2n-1}=\frac{1}{2}$ and prove that $p_{2n+1}=\frac{1}{2}$. Indeed,

$$\begin{aligned}
p_{2n+1}&=\prod_{k=1}^{2n+1}\left(1+\frac{(-1)^n}{n+1}\right)\\
&=\left[\prod_{k=1}^{2n-1}\left(1+\frac{(-1)^n}{n+1}\right)\right]\cdot\left(1+\frac{(-1)^{2n}}{2n+1}\right)\left(1+\frac{(-1)^{2n+1}}{2n+2}\right)\\
&=p_{2n-1}\cdot\left(1+\frac{1}{2n+1}\right)\left(1-\frac{1}{2n+2}\right)\\
&=\frac{1}{2}\cdot\left(\frac{2n+2}{2n+1}\right)\left(\frac{2n+1}{2n+2}\right)\\
&=\frac{1}{2}.
\end{aligned}$$

We conclude by mathematical induction that $\forall n\in\mathbb{N}\,(p_{2n-1}=\frac{1}{2})$.

(d) We may use mathematical induction and produce a proof similar to the one from the previous part. Alternatively, we may use our result from the previous part to conclude

$$\begin{aligned} p_{2n} &= \prod_{k=1}^{2n} \left(1 + \frac{(-1)^n}{n+1}\right) \\ &= \left[\prod_{k=1}^{2n-1} \left(1 + \frac{(-1)^n}{n+1}\right)\right] \cdot \left(1 + \frac{(-1)^{2n}}{2n+1}\right) \\ &= p_{2n-1} \cdot \left(1 + \frac{1}{2n+1}\right) \\ &= \frac{1}{2}\left(\frac{2n+2}{2n+1}\right) \\ &= \frac{n+1}{2n+1}. \end{aligned}$$

(e) Since $\lim_{n\to\infty} \frac{1}{2} = \lim_{n\to\infty} \frac{n+1}{2n+1} = \frac{1}{2}$, the even and the odd terms both converge to the same number so that[22]

$$\lim_{n\to\infty} p_n = \prod_{k=1}^{\infty} \left(1 + \frac{(-1)^n}{n+1}\right) = \frac{1}{2}.$$

(Exercise on page 90.)

[22] If you'd like to practice calculus proofs, we encourage you to use the definition of the limit of a sequence to prove this observation: If $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are sequences which converge to the same limit, then the "alternating sequence" $a_1, b_1, a_2, b_2, a_3, b_3, \ldots$ also converges to that limit.

Solution for Exercise 78 (Recurrence)**.**

Let $P(n)$ be the predicate $a_n < 2^n$. We prove $\forall n \in \mathbb{N}.P(n)$ via strong induction.

Base cases: Note that $a_1 = 1 < 2^1$ and $a_2 = 3 < 2^2$.

Inductive step: Let $n \in \mathbb{N}$ be arbitrary and suppose for all natural numbers $m < n$ we have $P(m)$.

If $n \leq 2$ then the base cases prove $P(n)$. Otherwise, $n \geq 3$ so that $a_n = a_{n-1} + a_{n-2}$. Since $n \geq 3$, we know that $n-1, n-2$ are natural numbers, and since $n-1, n-2 < n$ the inductive hypothesis asserts $P(n-1), P(n-2)$. That is, $a_{n-1} < 2^{n-1}$ and $a_{n-2} < 2^{n-2}$. We therefore have

$$\begin{aligned} a_n &= a_{n-1} + a_{n-2} \\ &< 2^{n-1} + 2^{n-2} \\ &= 3 \cdot 2^{n-2} \\ &< 4 \cdot 2^{n-2} \\ &= 2^n. \end{aligned}$$

That is, $a_n < 2^n$, which is $P(n)$. By mathematical induction we conclude that $\forall n \in \mathbb{N}.P(n)$.

(Exercise on page 91.)

Solution for Exercise 79 (Remainder modulo 3).
Let $P(n)$ be the predicate

$$\exists q \in \mathbb{Z}_{\geq 0}. \exists r \in \{0,1,2\}.(n = 3q + r).$$

We prove $\forall n \in \mathbb{N}.P(n)$ be strong induction.
Base cases: Note that

$$\begin{aligned}1 &= 3 \cdot 0 + 1,\\ 2 &= 3 \cdot 0 + 2,\\ 3 &= 3 \cdot 1 + 0.\end{aligned}$$

Inductive step: Let $n \in \mathbb{N}$ be arbitrary and suppose for every natural number $m < n$ we have $P(m)$. If $n \leq 3$, then the base cases show $P(n)$. Suppose therefore that $n \geq 4$. Then $n - 3$ is a natural number and since $n - 3 < n$ we have by the inductive hypothesis $P(n - 3)$. Therefore, there exists some $q \in \mathbb{Z}_{\geq 0}$ and $r \in \{0,1,2\}$ such that

$$n - 3 = 3q + r.$$

Then,

$$n = n - 3 + 3 = 3q + r + 3 = 3(q + 1) + r.$$

Since $q \in \mathbb{Z}_{\geq 0}$ we have $q + 1 \in \mathbb{Z}_{\geq 0}$ and we also know that $r \in \{0,1,2\}$; this proves $P(n)$. By mathematical induction we conclude that $\forall n \in \mathbb{N}.P(n)$.

(Exercise on page 92.)

Solution for Exercise 80 (Making change)**.**

Let $P(n)$ be the predicate

$$\exists s, t, u \in \mathbb{Z}_{\geq 0}.n = 6s + 10t + 15u.$$

We prove $\forall n \in \mathbb{N}.[(n \geq 30) \implies P(n)]$ by strong induction on n.

Base cases:

$$\begin{aligned}
30 &= 6 \cdot 0 + 10 \cdot 0 + 15 \cdot 2, \\
31 &= 6 \cdot 1 + 10 \cdot 1 + 15 \cdot 1, \\
32 &= 6 \cdot 2 + 10 \cdot 2 + 15 \cdot 0, \\
33 &= 6 \cdot 3 + 10 \cdot 0 + 15 \cdot 1, \\
34 &= 6 \cdot 4 + 10 \cdot 1 + 15 \cdot 0, \\
35 &= 6 \cdot 0 + 10 \cdot 2 + 15 \cdot 1.
\end{aligned}$$

Inductive step: Let $n \geq 30$ be arbitrary and suppose every that for every m with $30 \leq m < n$ we have $P(m)$.

If $30 \leq n \leq 35$ then $P(n)$ by the base case. Otherwise, $n \geq 36$ so that $n - 6 \geq 30$ and we have $P(n-6)$ by the inductive hypothesis. Let $s, t, u \in \mathbb{Z}_{\geq 0}$ be nonnegative integers such that

$$n - 6 = 6s + 10t + 15u.$$

Then,

$$n = n - 6 + 6 = 6s + 10t + 15u + 6 = 6(s + 1) + 10t + 15u.$$

Since $s, t, u \in \mathbb{Z}_{\geq 0}$ we also have $s + 1, t, u \in \mathbb{Z}_{\geq 0}$, so this proves that $P(n)$. By mathematical induction we conclude that $\forall n \in \mathbb{N}.[(n \geq 30) \implies P(n)]$.

(Exercise on page 93.)

Solution for Exercise 81 (Fibonacci)**.**
Let $P(n)$ be the predicate

$$f_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}.$$

We prove $\forall n \in \mathbb{N}.P(n)$ via strong induction.
Base cases:

- For $n = 1$ we have
$$\frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}} = \frac{2\sqrt{5}}{2\sqrt{5}} = 1 = f_1.$$
- For $n = 2$ we have
$$\frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}} = \frac{4\sqrt{5}}{4\sqrt{5}} = 1 = f_2.$$

Inductive step: Let n be an arbitrary natural number and suppose that for every natural number m with $m < n$ we have $P(m)$.
If $n \leq 2$, then $P(n)$ by the base cases. Otherwise $n \geq 3$ and by definition of the Fibonacci sequence we have

$$f_n = f_{n-2} + f_{n-1}.$$

Since $n \geq 3$ we know that $n-1, n-2$ are natural numbers and since $n-1, n-2$, the inductive hypothesis asserts $P(n-1)$ and $P(n-2)$. That is,

$$f_{n-1} = \frac{(1+\sqrt{5})^{n-1} - (1-\sqrt{5})^{n-1}}{2^{n-1}\sqrt{5}}, \qquad f_{n-2} = \frac{(1+\sqrt{5})^{n-2} - (1-\sqrt{5})^{n-2}}{2^{n-2}\sqrt{5}}.$$

Therefore,

$$\begin{aligned}
f_n &= f_{n-1} + f_{n-2} \\
&= \frac{(1+\sqrt{5})^{n-1} - (1-\sqrt{5})^{n-1}}{2^{n-1}\sqrt{5}} + \frac{(1+\sqrt{5})^{n-2} - (1-\sqrt{5})^{n-2}}{2^{n-2}\sqrt{5}} \\
&= \frac{(1+\sqrt{5})^{n-1} - (1-\sqrt{5})^{n-1} + 2\left[(1+\sqrt{5})^{n-2} - (1-\sqrt{5})^{n-2}\right]}{2^{n-1}\sqrt{5}} \\
&= \frac{(1+\sqrt{5})^{n-2}(1+\sqrt{5}+2) - (1-\sqrt{5})^{n-2}(1-\sqrt{5}+2)}{2^{n-1}\sqrt{5}} \\
&= \frac{(1+\sqrt{5})^{n-2}\cdot\frac{6+2\sqrt{5}}{2} - (1-\sqrt{5})^{n-2}\cdot\frac{6-2\sqrt{5}}{2}}{2^{n-1}\sqrt{5}} \\
&= \frac{(1+\sqrt{5})^{n-2}(1+\sqrt{5})^2 - (1-\sqrt{5})^{n-2}(1-\sqrt{5})^2}{2^n\sqrt{5}} \\
&= \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}.
\end{aligned}$$

Which proves $P(n)$. By mathematical induction we conclude that $\forall n \in \mathbb{N}.P(n)$.

Helpful Tip!

You may have noticed by now that induction is an incredibly powerful tool for proving all sort of mathematical statements. However, it is spectacularly unhelpful in finding those statements in the first place! You might wonder how did one find such a "strange" formula for the Fibonacci sequence—one involving fractions and irrational numbers that "magically" cancel out to give only positive integers! Linear algebra (specifically, eigenvalues and eigenvectors) can be used to find closed-form formulas for many linear recurrences. Similarly, the formulas for the sum of integers, sum of squares, sum of cubes, and so on can be found most efficiently via telescoping. More complex sequences are studied via the theory of generating functions.

(Exercise on page 94.)

Solution for Exercise 82 (Divisibility).

Let $P(n)$ be the predicate $(a+b)|(a^{2n-1}+b^{2n-1})$. We prove $\forall n \in \mathbb{N}\, P(n)$ by strong induction.

Base cases: $P(1)$ asserts the obvious fact $(a+b)|(a+b)$, whereas $P(2)$ asserts $(a+b)|(a^3+b^3)$ which is true as $(a^3+b^3) = (a+b)(a^2-ab+b^2)$.

Inductive step: Let $n \in \mathbb{N}$ be arbitrary and suppose that for every natural number $m < n$ we have $P(m)$.

If $n \leq 2$, then $P(n)$ is proved in the base cases. Suppose therefore that $n \geq 3$.

In order to make use of the inductive hypothesis, we are looking to express $a^{2n-1}+b^{2n-1}$ in terms of $a^{2n-3}+b^{2n-3}$ and lower order powers. Note that

$$\begin{aligned} a^{2n-1}+b^{2n-1} &= (a^{2n-3}+b^{2n-3})(a^2+b^2) - (a^2b^{2n-3}+b^2a^{2n-3}) \\ &= (a^{2n-3}+b^{2n-3})(a^2+b^2) - a^2b^2(a^{2n-5}+b^{2n-5}). \end{aligned}$$

Now, $n \geq 3$ so that $n-1, n-2$ are natural number. Since $n-1, n-2 < n$ the inductive hypothesis asserts $P(n-1)$ and $P(n-2)$. That is, $(a+b)|(a^{2(n-1)-1}+b^{2(n-1)-1})$ and $(a+b)|(a^{2(n-2)-1}+b^{2(n-2)-1})$. Therefore, there exist some $k, k' \in \mathbb{N}$ such that

$$a^{2n-3}+b^{2n-3} = (a+b)k, \qquad a^{2n-5}+b^{2n-5} = (a+b)k'.$$

It now follows that

$$\begin{aligned} a^{2n-1}+b^{2n-1} &= (a^{2n-3}+b^{2n-3})(a^2+b^2) - a^2b^2(a^{2n-5}+b^{2n-5}) \\ &= (a+b)k(a^2+b^2) - a^2b^2(a+b)k' \\ &= (a+b)[(a^2+b^2)k - a^2b^2k']. \end{aligned}$$

Since $a, b, k, k' \in \mathbb{N}$ we know that $[(a^2+b^2)k - a^2b^2k'] \in \mathbb{Z}$ is an integer. Moreover, since $a, b \in \mathbb{N}$ we know that $a^{2n-1}+b^{2n-1} > 0$ so that we must have $[(a^2+b^2)k - a^2b^2k'] > 0$ is a positive integer, so that $[(a^2+b^2)k - a^2b^2k'] \in \mathbb{N}$. We conclude that $a^{2n-1}+b^{2n-1}$ is divisible by $a+b$ (with quotient $(a^2+b^2)k - a^2b^2k'$). This is $P(n)$.

By mathematical induction we conclude that $\forall n \in \mathbb{N}. P(n)$.

(Exercise on page 95.)

Solution for Exercise 83 (Maximum and Minimum)**.**

Let us address each part in turn.

(a) The minimum of $\mathbb{N}$ is the number 1, since every natural number $n \in \mathbb{N}$ satisfies $1 \leq n$.

On the other hand, $\mathbb{N}$ does not have a maximum. We can prove this fact as follows: suppose towards contradiction $\mathbb{N}$ does have a maximum element m. Then, by the definition of a maximum, $n \leq m$ for every $n \in \mathbb{N}$. Now consider $n = m + 1$, clearly $n = m + 1 > m$, which contradicts that m is a maximum. Therefore, $\mathbb{N}$ does not have a maximum element.

(b) The set of integers $\mathbb{Z}$ does not have a maximum or a minimum. The proof that $\mathbb{Z}$ does not have a maximum is the same as the one above that $\mathbb{N}$ does not have a maximum.

An analogous proof shows that $\mathbb{Z}$ does not have a minimum: suppose towards contradiction $\mathbb{Z}$ has a minimum m. Then, by the definition of a minimum, for every $z \in \mathbb{Z}$, we have $m \leq z$. Consider the integer $m - 1$. Clearly, $m - 1 < m$, so that $m \leq m - 1$ is not true. This contradicts that for every $z \in \mathbb{Z}$, we have $m \leq z$, and that m is a minimum for $\mathbb{Z}$. It follows that $\mathbb{Z}$ does not have a minimum.

(c) $\emptyset$. Since the empty set contains no elements, it does not contain a maximum or a minimum.

(d) $A = \{n \in \mathbb{N} : n \text{ is a multiple of } 3\}$. Let's look at the elements of the set A,

$$A = \{3, 6, 9, 12, 15, 18, \dots, 3k, \dots\} = \{3k : k \in \mathbb{N}\}.$$

The minimum of A is 3. To prove this, we need to show that $3 \leq 3k$ for every $k \in \mathbb{N}$. This follows from the fact that $1 \leq k$ for every $k \in \mathbb{N}$ so multiplying by (the positive integer) 3 we obtain $3 \leq 3k$ for every $k \in \mathbb{N}$.

On the other hand, A does not have a maximum. Suppose towards contradiction A has a maximum element m. Then by the definition of a maximum element, $n \leq m$ for every $n \in A$. Now, the elements of A are natural numbers that are multiples of 3, so we must have $m = 3k$ for some natural number k. Consider $n = 3(k + 1)$. We have $n = 3(k + 1) = 3k + 3 > 3k = m$, which contradicts that m is a maximum. Therefore, A does not have a maximum element.

(e) $B = \{z \in \mathbb{Z} : z > 11\}$. Let's look at the elements of the set B,

$$B = \{12, 13, 14, 15, 16, \dots\}.$$

The minimum of set B is 12 because $12 \leq z$ for every $z \in B$.

On the other hand, B does not have a maximum. Suppose towards contradiction B does have a maximum element m. Then by definition, $n \leq m$ for every $n \in B$. Now, the elements of B are natural numbers that are strictly greater than 11, so we must have $m > 11$. Consider $n = m + 1$. Since $m > 11$, we also have $m+1 > 11$, so $m+1 \in B$. However, $n = m+1 > m$, which contradicts that m is a maximum. Therefore, B does not have a maximum element.

(f) $C = \{r \in \mathbb{R} : 0 < r < 1\} = (0, 1)$. The set C does not have a maximum or a minimum.

Assume for contradiction C has a maximum x. Then $0 < x < 1$ (since $x \in C$), and for every $y \in (0, 1)$ we have $y \leq x$ (since x is a maximum). Consider $r = \frac{x+1}{2}$. Since $0 < x < 1$, we have

$$0 < \frac{0+1}{2} < \frac{x+1}{2} < \frac{1+1}{2} = 1$$

proving that $r \in C$. But $r = \frac{x+1}{2} > x$ (since $x + 1 > 2x \iff x < 1$)[23], which contradicts that x is a maximum of C. Therefore, C has no maximum.

The proof that C has no minimum element is analogous: suppose towards contradiction C has a minimum m. Then $0 < m < 1$ and for every $y \in (0, 1)$ we have $m \leq y$. Consider $q = \frac{m}{2}$. We have $q < m$ (since $m < 2m \iff m > 0$). Also, we have $0 < m < 1$, and we can divide this inequality by (the positive integer) 2 to get $0 < \frac{m}{2} < \frac{1}{2}$, so that $q \in C$. But $q < m$, which contradicts the minimality of m. Therefore, C does not have a minimum.

(g) $D = \{x \in \mathbb{R} : 0 \leq x \leq 1\} = [0, 1]$. The minimum of D is 0, and its maximum is 1. This follows directly from the definition of the set D: for every element $x \in D$, we have $0 \leq x$ as well as $x \leq 1$, and $0, 1 \in D$.

(Exercise on page 97.)

[23] You can see this fact clearly on the number line, since r is the midpoint between x and 1.

Solution for Exercise 84 (Spot the error)**.**

Recall the Well-Ordering Principle states that every nonempty subset of the natural numbers has a least element.

(a) The mistake here is that the subset must be **nonempty**. As we saw in Exercise 83, the empty set does not have a least element.

(b) The mistake here is in the set, it should be the natural numbers, and not the integers. For example, the set $\mathbb{Z}$, which is a nonempty subset of itself, does not have a least element (as was shown in Exercise 83).

(c) The mistake here is in the 'greatest element' part. For example, the set $\mathbb{N}$, which is a nonempty subset of itself, does not have a greatest element (as was shown in Exercise 83).

(Exercise on page 98.)

Solution for Exercise 85 (Well-Ordering from Induction).

(a) We prove $\forall n \in \mathbb{N}.P(n)$ by (complete) induction.

Base case: We prove $P(1)$. If $1 \in S$, then 1 would be a least element of S. This is because $S \subseteq \mathbb{N}$ and for every natural number n we have $1 \leq n$. Therefore, for every $s \in S$ we have $1 \leq s$. Since S has no minimum element, we must have $1 \notin S$, which is $P(1)$.

Inductive step: Let $n \in \mathbb{N}$ be arbitrary and suppose for all natural numbers $k < n$ we have $P(k)$. Suppose towards contradiction that $n \in S$. Then n would be a least element. This is because $k < n \implies P(k)$ so that $(k < n) \implies (k \notin S)$. The contrapositive is $(k \in S) \implies (k \geq n)$ proving that n is minimum. However, this contradicts the assumption that S has no minimal element. Therefore, $n \notin S$, which is $P(n)$.

We conclude by mathematical induction that $\forall n \in \mathbb{N}.P(n)$.

(b) We have supposed that S does not have a least element and proved that $\forall n \in \mathbb{N}.P(n)$, i.e. no natural number is an element of S. Since S is assumed to be a subset of $\mathbb{N}$, it must be empty. But this contradicts the assumption that S is a nonempty subset of $\mathbb{N}$. This contradiction shows that our supposition was false: S must have a least element.

In summary, we supposed that the well-ordering principle fails; that is, there exists a nonempty subset $S \subseteq \mathbb{N}$ with no least element. We then derived a contradiction using the principle of induction. Therefore, our supposition must be false. Thus, every nonempty subset of $\mathbb{N}$ must have a least element, which is exactly the Well-Ordering Principle.

(Exercise on page 99.)

Solution for Exercise 86 (Induction from Well-Ordering)**.**

Let $S \subset \mathbb{N}$ such that

- $1 \in S$
- $\forall n \in \mathbb{N}.[(n \in S) \implies (n+1 \in S)]$.

Suppose towards contradiction $S \neq \mathbb{N}$ and consider $S^c = \{n \in \mathbb{N} : n \notin S\}$. Since $S \neq \mathbb{N}$ we know there exists some $n \in \mathbb{N}$ such that $n \notin S$, so that S^c is nonempty.

Since S^c is a nonempty subset of $\mathbb{N}$, the well-ordering principle guarantees S^c has some minimal element, $m \in S^c$.

Note that $m \neq 1$, since $1 \in S$ and therefore $1 \notin S^c$. Therefore, $m - 1$ is a natural number. Since m is the minimal element of S^c, we have $m-1 \notin S^c$ so $m-1 \in S$. By assumption, $(m-1 \in S) \implies m \in S$, so we conclude that $m \in S$, contradicting the fact that $m \notin S$.

We have suppose $S \neq \mathbb{N}$ and derived a contradiction using the well-ordering principle. This contradiction proves that $S \neq \mathbb{N}$ is false, i.e. $S = \mathbb{N}$.

(Exercise on page 100.)

Solution for Exercise 87 (Using the Well-Ordering Principle)**.**

Let $P(n)$ be the predicate

$$\sum_{k=1}^{n} 2k = n(n+1).$$

We wish to prove $\forall n \in \mathbb{N}.P(n)$. Consider the set of possible counter-examples:

$$S = \{n \in \mathbb{N} : \neg P(n)\}.$$

We shall prove $S = \emptyset$, as this implies $\forall n \in \mathbb{N}.P(n)$.

Suppose towards contradiction $S \neq \emptyset$. By the well-ordering principle, S has a least element $m \in S$.

Note that $m \neq 1$ since $P(1)$ is true: $2 = 1(1+1)$. Therefore $m > 1$, so $m-1$ is a natural number. Since $m-1 < m$ and m is the minimal element of S, we must have $m-1 \notin S$, i.e. $P(m-1)$.

Therefore,

$$\sum_{k=1}^{m-1} 2k = (m-1)m$$

and adding $2m$ to both sides of the equality we have

$$\sum_{k=1}^{m} 2k = (m-1)m + 2m = m(m-1+2) = m(m+1)$$

which proves $P(m)$. Since $P(m)$ is true we have $m \notin S$, contradicting the definition of m as the minimal element of S.

We assumed that S is nonempty and arrived at a contradiction, therefore we conclude that S is empty and $\forall n \in \mathbb{N}.P(n)$.

Helpful Tip!

Note that the main step here (going from $P(m-1)$ to $P(m)$) is the same as in an inductive proof.

(Exercise on page 101.)

Solution for Exercise 88 (Division with remainder)**.**

(a) We will show that $n \in S$. Since $n \in \mathbb{N}$ we have $n \in \mathbb{Z}_{\geq 0}$, so that it is an element of the domain. Choosing $q = 0$ we have $n = n - qm$, so that it satisfies the defining condition of S. This proves that $n \in S$ and therefore $S \neq \emptyset$.

(b) Let r be the minimal element of S. By the definition of S, $r \in \mathbb{Z}_{\geq 0}$ so that $r \geq 0$. It remains to prove that $r \leq m - 1$.

Suppose towards contradiction that $r \geq m$. By the definition of S, there exists some $q \in \mathbb{Z}_{\geq 0}$ such that $r = n - qm$. Then $n - qm \geq m$ and therefore $n - (q + 1)m \geq 0$ is a nonnegative integer. Now, $q + 1 \in \mathbb{Z}_{\geq 0}$ so that the nonnegative integer $n - (q + 1)m$ satisfies the defining property of S and $n - (q + 1)m \in S$. But $n - (q + 1)m < n - qm$ (since $m \in \mathbb{N}$ and so $m > 0$), contradicting the minimality of r. This contradiction proves that $r \leq m - 1$.

(c) Since S is a nonempty subset of the integers which is bounded below, the well-ordering principle guarantees it has a minimum element. Let $r \in S$ be the minimum element. Then $\exists q \in \mathbb{Z}_{\geq 0}$ such that $r = n - qm$ and therefore $n = qm + r$. We have shown in the previous part that $r \in \{0, \ldots, m - 1\}$, and so we conclude that there exist $q \in \mathbb{Z}_{\geq 0}$ and $r \in \{0, 1, \ldots, m - 1\}$ such that $n = qm + r$.

(d) Suppose $q, q' \in \mathbb{Z}_{\geq 0}$ and $r, r' \in \{0, 1, \ldots, m - 1\}$ are such that $n = qm + r = q'm + r'$.

Suppose towards contradiction that $q \neq q'$ and assume without loss of generality that $q > q'$. Then $(q - q')m = (r' - r)$ and since $q - q' > 0$ the left side is at least m. But the right side is at most $m - 1$ (since $0 \leq r, r' \leq m - 1$ implies $r' - r \leq (m - 1) - 0$), which is a contradiction. This contradiction proves that $q = q'$, and since $qm + r = q'm + r'$ we conclude that $r = r'$ as well.

(Exercise on page 102.)

Solution for Exercise 89 (Spot the error II).

The recursive formula $f_n = f_{n-1} + f_{n-2}$ is only valid for $n \geq 3$. If we could show that $s \geq 3$, we would indeed have a contradiction. But in fact $s = 1$ is the minimal element of S, since $f_1 = 1$ is odd.

Helpful Tip!

Compare to false inductive proofs which do not verify the base case, as the one from the handout on induction.

(Exercise on page 103.)

Solution for Exercise 90 (Roundabout)**.**

Let S be the set of possible round-trips. We first prove that S is nonempty.

Start at any city and keep going. It is always possible to exit the city because it is possible to reach from any city to any other city. Since there are a finite number of cities eventually your tour will take you to a city you've visited. The portion of the tour between the first time you've exited that city and the first time you returned to that city is a round-trip.

Since S is nonempty, it has a round-trip R of minimum length by the well-ordering principle. We claim that such a round trip will not visit the same city twice (except for the starting and ending city). Indeed, if there is a city in the middle which is visited twice then the portion of the tour between the two visitations of this city is a round-trip shorter than R, a contradiction. Similarly, if the starting city is visited again before the end, then that portion of the tour is a round-trip shorter than R, a contradiction. We conclude that R is a round-trip which doesn't visit any city more than once (and the starting and ending city exactly twice).

(Exercise on page 104.)

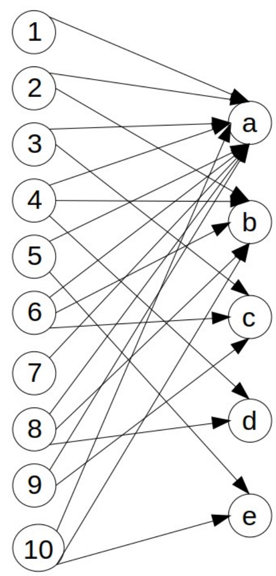
Solution for Exercise 91 (Describing Relations).

(a) The question states that R is a relation *from X to Y*. Therefore, $R \subseteq X \times Y$.

(b) Following the description of the set, we have

$$R = \{(1,a);(2,a);(3,a);(4,a);(5,a);(6,a);(7,a);(8,a);(9,a);(10,a); \\ (2,b);(4,b);(6,b);(8,b);(10,b);(3,c);(6,c);(9,c);(4,d);(8,d);(5,e);(10,e)\}$$

(c) The digraph below is one way to illustrate the relation. As you can see, it becomes hard to read quite quickly. It does have some advantages, like showing at a glance which elements are related to many other elements.



(d) For ease of readability, we left the 0-cells empty. We get the following table.

	a	b	c	d	e
1	1				
2	1	1			
3	1		1		
4	1	1		1	
5	1				1
6	1	1	1		
7	1				
8	1	1		1	
9	1		1		
10	1	1			1

(Exercise on page 105.)

Solution for Exercise 92 (Properties of Relations)**.**

Here is a summary of the properties of each relation:

	reflexive	symmetric	transitive
C	no	no	yes
D	yes	yes	yes
N	no	yes	no
S	yes	yes	no

We now explain each of our answers:

- The relation C of "having taken more courses".
  - No student has taken more courses than themselves, so the relation is not reflexive.
  - If a has taken more courses than b, then b has taken *less* courses than a, not more—so the relation is not symmetric.
  - If a has taken more courses than b and b has taken more courses than c, we can conclude a has taken more courses than c, so the relation is symmetric.
- The relation D of "being in the same degree program".
  - Each student is in the same degree program as themselves, so the relation is reflexive.
  - If a is in the same degree program as b, then b is in the same degree program as a—so the relation is symmetric.
  - If a is in the same degree program as b and b is in the same degree program as c, then a and c are in the same degree program—so the relation is transitive.[24]
- The relation N of "having no courses in common this semester".
  - Each student has (all of their) courses in common with themselves, so the relation is not reflexive.
  - If a has no courses in common with b, then b has no courses in common with a—so the relation is symmetric.
  - It is possible for a and b to have no courses in common, for b and c to have no courses in common, and for a and c to have at least one course (or even all of them) in common. For example, if a and c are taking Linear Algebra and Analysis, while b is taking Combinatorics and Geometry. In conclusion, the relation is not (in general) transitive.
- The relation S of "having at least one course in common".
  - Each student has (all of their) courses in common with themselves, so the relation is reflexive.
  - If a has at least one course in common with b, then b has at least one course in common with a (the same course!)—so the relation is symmetric.
  - It is possible for a and b to have a course in common, for b and c to have a different course in common, and for a and c to have no courses in common. For example, if a is taking Linear Algebra and Analysis, b is taking Linear Algebra and Combinatorics, and c is taking Combinatorics and Geometry. In conclusion, the relation is not (in general) transitive.

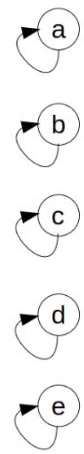
(Exercise on page 106.)

[24] This is assuming that each student is only ever enrolled in a single degree program at a time. If you have a different assumption your answer would be different, but it's important that you explain your assumptions.

Solution for Exercise 93 (Properties of Relations)**.**

(a) If R is reflexive, it means that every element is related to itself. We modify the logical table and the digraph to show this fact.

	a	b	c	d	e
a	1				
b		1			
c			1		
d				1	
e					1



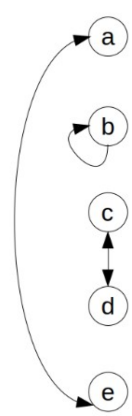
(b) Let us address each description in turn.

- If the relation is described as a set, for each $z \in A$ we would have to check that (z, z) is in the set.
- If the relation is described using a digraph, we need to check that each node has an arrow to itself.
- If the relation is described as a logical table, we need to check that the main diagonal is all 1.

Perhaps the easiest description to check is the logical table, as we can see the main diagonal at a glance. The digraph can be complicated with many arrows which can make it difficult to tell at a glance whether the relation is reflexive. The set description is the worst, as we would need to search for each element of the form (z, z) in a potentially very large list.

(c) Since cSd and the relation is symmetric, we must also have dSc. Similarly, from eSa we deduce aSe.

	a	b	c	d	e
a					1
b		1			
c				1	
d			1		
e	1				



(d) Let us again address each description in turn.

- In the set description we need to check that whenever (y, z) is an element of the set, so is (z, y).
- With the digraph description, we need to check that each arrow (except for self-loops!) is bidirectional.
- With the table description, we need to check that the table is symmetric about the main diagonal. That is, whenever the cell in row i column j is 1, so is the cell in row j column i.

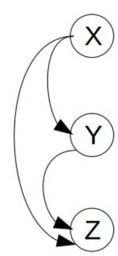
The logical table is usually the easiest way to determine whether a relation is symmetric; most of the time it is straightforward to tell at a glance whether the table is symmetric or not. The graph description can get cluttered quickly and the set description is too lengthy.

(e) Determining whether a relation is transitive is usually tricky, because it cannot be done "at a glance" using any of the three common descriptions[25]. The set description is the worse, as the information is not organized in any convenient way. The table description is a little bit better.

- We examine each row of the table. Suppose we are in row i, if we see a 1 in column j, then we go to row j and check that if we see a 1 in column k, there is also a 1 in row i column k.

The most convenient description to check transitivity is perhaps the digraph.

- For each node X and each out-arrow to any another node Y, we check that for each out-arrow from Y to Z there is also an arrow from X to Z. Below is a simplified illustration of this situation.



(Exercise on page 107.)

[25]Most often, the definition of the relation is used directly to prove that it is transitive, saving a lot of work (sometimes even an infinite amount).

Solution for Exercise 94 (Counting Relations).

(a) If R is a relation on the set $\{1,2,3\}$, this means that R is a subset of $\{1,2,3\}\times\{1,2,3\}$ (which can also be denoted $\{1,2,3\}^2$).

(b) Any subset of $\{1,2,3\}\times\{1,2,3\}$ is a relation on $\{1,2,3\}$. Since $\{1,2,3\}\times\{1,2,3\}$ has precisely 9 elements, there are exactly $2^9 = 512$ different possible relations on $\{1,2,3\}$ (these include the empty relation—nothing is related to anything else—and the trivial relation—every element is related to every other element).

In general, the set $\{1,2,\ldots,n\}^2$ has n^2 elements, and so the number of relations on $\{1,2,\ldots,n\}$ is just the number of possible subsets of $\{1,2,\ldots,n\}^2$, which is 2^{n^2}.

Helpful Tip!

Note that $2^{n^2} = 2^{\left(n^2\right)}$ is very different from $\left(2^n\right)^2 = 2^{2n}$. For example, for $n = 3$, we have already seen that $2^{3^2} = 2^9 = 512$, whereas $\left(2^3\right)^2 = 8^2 = 64$.

(c) If R is a reflexive relation on $\{1,2,3\}$, every element must be related to itself; that is, $1R1$, $2R2$, and $3R3$. Therefore, R must contain the elements $(1,1)$, $(2,2)$, and $(3,3)$ (and R may or may not contain other elements).

(d) We have already seen that any reflexive relation R on $\{1,2,3\}$ must contain the three elements $(1,1)$, $(2,2)$, and $(3,3)$. Any of the remaining 6 elements of the set $\{1,2,3\}\times\{1,2,3\}$ may or may not be an element of R. Therefore, the possible reflexive relations on $\{1,2,3\}$ are all of the form

$$\{(1,1),(2,2),(3,3)\}\cup S$$

where S is any subset of $\{(1,2),(1,3),(2,1),(2,3),(3,1),(3,2)\}$. Since there are $2^6 = 64$ possible such subsets, we see that there are 64 different reflexive relations on $\{1,2,3\}$.

In general, a reflexive relation on $\{1,2,\ldots,n\}$ is of the form

$$\{(1,1),(2,2),\ldots,(n,n)\}\cup S$$

where S is a subset of $T := \{1,2,\ldots,n\}^2 \setminus \{(1,1),(2,2),\ldots,(n,n)\}$. This bigger set T has $n^2 - n$ elements and so 2^{n^2-n} subsets. Therefore, there are 2^{n^2-n} different reflexive relations on $\{1,2,\ldots,n\}$.

(e) We have seen that the possible reflexive relations on $\{1,2,3\}$ are all of the form

$$\{(1,1),(2,2),(3,3)\}\cup S$$

where S is any subset of $\{(1,2),(1,3),(2,1),(2,3),(3,1),(3,2)\}$. If the relation is also required to be symmetric then $(1,2)$ is an element of the relation if and only if $(2,1)$ is. Similarly, $(1,3)$ is an element of the relation if and only if $(3,1)$ is; the same is true for the pair $(2,3)$ and $(3,2)$. Therefore, we have three choices: whether to include both $(1,2)$ and $(2,1)$; whether to include both $(1,3)$ and $(3,1)$; whether to include both $(2,3)$ and $(3,2)$. There are therefore $2^3 = 8$ different relations on $\{1,2,3\}$ that are both reflexive and symmetric.

Generalizing, we have seen that a reflexive relation on $\{1,2,\ldots,n\}$ is of the form

$$\{(1,1),(2,2),\ldots,(n,n)\}\cup S$$

where S is a subset of $T := \{1, 2, \ldots, n\}^2 \setminus \{(1, 1), (2, 2), \ldots, (n, n)\}$. If the relation is also symmetric, then each $(a, b) \in T$ is an element of the relation if and only if (b, a) is as well. Therefore, we have $(n^2 - n)/2$ choices for a total of $2^{(n^2-n)/2}$ different relations on $\{1, 2, \ldots, n\}$ that are both reflexive and symmetric.

Note: We have seen in Chapter 2 that $n^2 - n = n(n - 1)$ is an even integer for every $n \in \mathbb{N}$, and we are using this property here! (In a sense, this can be seen as another proof of this property.)

(f) There are precisely 5 different relations on $\{1, 2, 3\}$ that are at the same time reflexive, symmetric, and transitive (i.e., equivalence relations). Indeed, we have seen that there are only 8 different relations on $\{1, 2, 3\}$ that are both reflexive and symmetric:

$$\begin{aligned}
R_1 &= \{(1, 1), (2, 2), (3, 3)\};\\
R_2 &= \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\};\\
R_3 &= \{(1, 1), (2, 2), (3, 3), (1, 3), (3, 1)\};\\
R_4 &= \{(1, 1), (2, 2), (3, 3), (2, 3), (3, 2)\};\\
R_5 &= \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (1, 3), (3, 1)\};\\
R_6 &= \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (2, 3), (3, 2)\};\\
R_7 &= \{(1, 1), (2, 2), (3, 3), (1, 3), (3, 1), (2, 3), (3, 2)\};\\
R_8 &= \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (1, 3), (3, 1), (2, 3), (3, 2)\}.
\end{aligned}$$

Except for R_5, R_6, and R_7, all of these relations are transitive. Each of R_5, R_6, and R_7 fails to be transitive for analogous reasons:

- For R_5 note that $3R1$ and $1R2$ would imply $3R2$, which is not the case;
- For R_6 note that $3R2$ and $2R1$ would imply $3R1$, which is not the case;
- For R_7 note that $1R3$ and $3R2$ would imply $1R2$ which is not the case.

(Exercise on page 108.)

Solution for Exercise 95 (Weak ordering)**.**

In each case we must prove that the relation is reflexive, antisymmetric, and transitive.

(a) We prove that $\leq$ is a weak ordering on $\mathbb{N}$.

- For every $n \in \mathbb{N}$ we have $n \leq n$, so $\leq$ is reflexive.
- Let $a, b \in \mathbb{N}$ be arbitrary and suppose $a \leq b$ and $b \leq a$. Then $a = b$. Since $a, b \in \mathbb{N}$ were arbitrary, this proves that $\leq$ is antisymmetric.
- Let $a, b, c \in \mathbb{N}$ be arbitrary and suppose $a \leq b$ and $b \leq c$, then we have $a \leq c$. Since $a, b, c \in \mathbb{N}$ were arbitrary, this proves that $\leq$ is transitive.

(b) We prove that $\subseteq$ is a weak ordering on $\mathcal{P}(\{1, 2, 3\})$.

- Every set is a subset of itself. That is, if $S \in \mathcal{P}(\{1, 2, 3\})$ then $S \subseteq S$. This proves that $\subseteq$ is reflexive.
- Suppose $A, B \in \mathcal{P}(\{1, 2, 3\})$ are such that $A \subseteq B$ and $B \subseteq A$. This is precisely the *definition* of set equality, so $A = B$. Since $A, B \in \mathcal{P}(\{1, 2, 3\})$ were arbitrary, this proves that $\subseteq$ is antisymmetric.
- Let $A, B, C \in \mathcal{P}$ be arbitrary and suppose $A \subseteq B$ and $B \subseteq C$. Let us prove that $A \subseteq C$, which would show that $\subseteq$ is transitive.

  Towards that end, let $a \in A$ be arbitrary. Since $A \subseteq B$, we have $a \in A \implies a \in B$. Since $B \subseteq C$ we have for any element in our universe $x \in B \implies x \in C$. In particular, $a \in B \implies a \in C$. Since a was an arbitrary element of A, this proves that for every element in our universe $a \in A \implies a \in C$, i.e. $A \subseteq C$.

(c) We prove that the divisibility relation is a weak ordering on $\mathbb{N}$.

- For every $n \in \mathbb{N}$ we have $n = n \cdot 1$, so $n|n$. This proves that n is reflexive.
- Let $a, b \in \mathbb{N}$ be arbitrary and suppose $a|b$ and $b|a$. Then $b = ak$ for some $k \in \mathbb{N}$ and $a = bk'$ for some $k' \in \mathbb{N}$. Since $k \in \mathbb{N}$, we know that $k \geq 1$ so that $b = ak \geq a$. Similarly, from $k' \geq 1$ we deduce $a = bk' \geq b$. Since $a \leq b$ and $b \leq a$, we conclude that $a = b$. This proves that the divisibility relation is antisymmetric.
- Let $a, b, c \in \mathbb{N}$ be arbitrary and suppose $a|b$ and $b|c$. Then there is some $k, k' \in \mathbb{N}$ such that $b = ak$ and $c = bk'$. Then $c = bk' = akk'$. Since $kk' \in \mathbb{N}$, this proves that $a|c$. Therefore, the divisibility relation is transitive.

(Exercise on page 109.)

Solution for Exercise 96 (Strict ordering).

(a) We prove that $<$ is a strict ordering on $\mathbb{N}$.

- Suppose $a, b \in \mathbb{N}$ are such that $a < b$. Then $\neg(b < a)$. This proves that $<$ is asymmetric.
- Suppose $a, b \in \mathbb{N}$ are such that $a < b$ and $b < c$. Then $a < c$. This proves that $<$ is transitive.

(b) Suppose R is a strict ordering on A and assume for contradiction that there is some $a \in A$ such that aRa. By the asymmetry of R (with $b = a$ in the definition), we conclude $\neg(aRa)$, which is a contradiction. This contradiction proves that there is no $a \in A$ such that aRa.

(c) We prove that S is asymmetric and transitive.

- Assume for contradiction that for some $a, b \in A$ we have aSb and bSa. By the definition of S this means aRb and $a \neq b$, and bRa and $a \neq b$. Since R is a weak ordering on A we know that R is antisymmetric. From aRb and bRa we conclude $a = b$, contradicting the assumption that $a \neq b$. This contradiction shows that if aSb then we must have $\neg(bSa)$. That is, S is asymmetric.
- Let $a, b, c \in A$ be arbitrary and suppose aSb and bSc. By the definition of S we have aRb and $a \neq b$ and bRc and $b \neq c$. Since R is a weak ordering on A we know that R is transitive, so from aRb and bRc we conclude aRc. It remains to prove that $a \neq c$.

  Assume for contradiction that $a = c$. Then from aSb and bSc we conclude aSb and bSa, contradicting the fact that S is asymmetric (which we have already proved above). Therefore $a \neq c$, and we have already shown aRc, so by the definition of S we have aSc. This proves that S is transitive.

(d) We prove that R is reflexive, antisymmetric, and transitive.

- Let $a \in A$ be arbitrary. Since $a = a$ we have aSa or $a = a$, so by the definition of R we conclude aRa.
- Let $a, b \in A$ and suppose aRb and bRa. By the definition of R we have aSb or $a = b$ as well as bSa or $a = b$. Since S is asymmetric it is not possible to have both aSb and bSa, so we conclude that $a = b$.
- Let $a, b, c \in A$ and suppose aRb and bRc. By the definition of R we have aSb or $a = b$ as well as bSc or $b = c$.
  - Suppose first that aSb. If bSc then by the transitivity of S we have aSc. On the other hand, if $b = c$ then from aSb we conclude aSc. Either way aSc so that aSc or $a = c$ and by the definition of R we have aRc.
  - Suppose next that $a = b$. If bSc then we conclude aSc. On the other hand, if $b = c$ we conclude $a = c$. Either way we have aSc or $a = c$ so by the definition of R we conclude aRc.

  We have shown aRc. Since $a, b, c \in A$ were arbitrary, this proves that R is transitive.

(Exercise on page 110.)

Solution for Exercise 97 (Real-world relations).

Recall that the properties of these relations were described in the previous handout. Out of all of these relations, only D "being in the same degree program" is reflexive, symmetric, and transitive and thus an equivalence relation[26]. The equivalence classes are the collection of University of Toronto students enrolled in each degree program.

(Exercise on page 111.)

[26]Under the assumption that each student is enrolled in only one degree program at a time.

Solution for Exercise 98 (String length)**.**

Let's check each property in turn.

- Reflexivity: Any word has the same number of letters as itself. Therefore, the relation is reflexive.
- Symmetry: Suppose $w_1 \sim w_2$, so w_1 has the same number of letters as w_2. This means that w_2 has the same number of letters as w_1, so that $w_2 \sim w_1$. This proves the relation is symmetric.
- Transitivity: Suppose $w_1 \sim w_2$ and $w_2 \sim w_3$. Since $w_1 \sim w_2$, we know that w_1 has the same number of letters as w_2, say ℓ letters. Since $w_2 \sim w_3$, we know that w_2 has the same number of letters as w_3. Since w_2 has ℓ letters and w_3 has the same number of letters, we know that w_3 also has ℓ letters. Then w_1 has the same number of letters as w_3 (namely, ℓ letters), so that $w_1 \sim w_3$. This proves that the relation is transitive.

Since $\sim$ is reflexive, symmetric, and transitive, it is an equivalence relation.

For each $n \in \mathbb{N}$, as long as there is at least one word in English with n letters, there is an equivalence class consisting of all the words of length n. For example, $[\texttt{cat}] = [\texttt{dog}]$ is the equivalence class of all words with 3 letters.

On the other hand, since there is no word in English with 10^{10} letters, there is no equivalence class for words of that length.

(Exercise on page 112.)

Solution for Exercise 99 (Digraphs)**.**

(a) The digraph does *not* represent an equivalence relation. The only property that fails is transitivity: we have cRa and aRd, but we do not have cRd.

(b) The digraph represents an equivalence relation. The relation is reflexive, symmetric, and (trivially) transitive.

The equivalence classes consists of the set of elements that are connected by arrows:

$$[a] = [d] = \{a, d\}, \qquad [b] = [c] = \{b, c\}.$$

(c) The digraph does *not* represent an equivalence relation. The only property that fails is transitivity: we have bRa and aRd, but we do not have bRd.

(Exercise on page 113.)

Solution for Exercise 100 (Common misconception)**.**

The mistake is in the assumption that *there exists* a $b \in X$ such that $a \sim b$.

A simple counterexample on $X = \{0, 1\}$ is given by the relation $\sim := \{(0, 0)\}$. The relation is symmetric and transitive. However, it is not reflexive since $1 \not\sim 1$. If we follow the proof with $a = 1$ in mind, we see that the line "for any b such that $a \sim b$" fails since there is no $b \in X$ such that $1 \sim b$.[27]

(Exercise on page 114.)

[27] The empty relation is another example, since it is vacuously symmetric and transitive. However, it is important to note that it is *not* the only exception; otherwise we would obtain a theorem along the lines "if a nonempty relation is symmetric and transitive . . . "

Solution for Exercise 101 (Multifunctional).

(a) Let R be a relation that is multifunctional, symmetric, and transitive. We wish to prove that R is an equivalence relation, so it remains to prove that R is reflexive.

Let $x \in X$ be arbitrary. Since R is multifunctional, there exists some $y \in X$ such that xRy. Since R is symmetric, from xRy we deduce yRx. Since R is transitive, from xRy and yRx we deduce xRx. This proves that R is reflexive, since $x \in X$ was arbitrary.

(b) The relation $\sim$ is

- **Not reflexive**. In fact it is, *irreflexive*: for any $A \in M_n(\mathbb{R})$ we have $A - A = O$ the all-zeroes matrix, which is never invertible. Therefore, $A \not\sim A$ for any $A \in M_n(R)$.
- **Symmetric**. Suppose $A, B \in M_n(\mathbb{R})$ satisfy $A \sim B$ so that $A - B$ is invertible with an inverse C. That is, $(A - B)C = I_n$ (the $n \times n$) identity matrix. Then $(B - A)(-C) = -(B - A)C = (A - B)C = I_n$, proving that $B - A$ is also invertible, so that $B \sim A$. This proves that $\sim$ is symmetric.
- **Multifunctional**. For any $A \in M_n(\mathbb{R})$ we have $A - I_n \in M_n(\mathbb{R})$ and $A \sim (A - I_n)$ because $A - (A - I_n) = I_n$ is invertible (with an inverse I_n). This proves that $\sim$ is multifunctional.
- **Not transitive**. We can now conclude that $\sim$ is not transitive. If $\sim$ were transitive it would be reflexive (since it is also multifunctional and symmetric), but we know that it is not reflexive, so it cannot be transitive.[28]

(Exercise on page 115.)

[28] A direct proof of this fact would in effect be an example of the theorem we have just proved in part (a). If you took the time to write a direct proof, compare your proof to the proof of the Theorem in part (a).

Solution for Exercise 102 (Remainders)**.**

(a) According to the Division with Remainder Theorem, there exist $q_a, q_b \in \mathbb{Z}_{\geq 0}$ and $r_a, r_b \in \{0, 1, \ldots, m-1\}$ such that

$$a = mq_a + r_a, \qquad b = mq_b + r_b.$$

Suppose now that aMb, so that $r_a = r_b$. We therefore have

$$b - a = m(q_b - q_a)$$

proving that $m|(b-a)$. (Note that since $b - a \geq 0$ we also have $q_b - q_a \geq 0$.)

Conversely, suppose that $m|(b-a)$ so there is some $q \in \mathbb{Z}_{\geq 0}$ such that $b - a = mq$. Then,

$$mq = b - a = m(q_b - q_a) + (r_b - r_a)$$

which we may rewrite as

$$m(q + q_a - q_b) = r_b - r_a.$$

Taking absolute value, we find that

$$m|q + q_a - q_b| = |r_b - r_a|.$$

On the other hand, $|r_b - r_a| < m$ (since $r_a, r_b \in \{0, 1, \ldots, m-1\}$) so that we must have $q + q_a - q_b = 0$ and consequently $r_b - r_a = 0$. That is, $r_a = r_b$, as we wanted to prove.

(b) We verify each defining property of an equivalence relation:

- Reflexivity. Since remainders are unique, every number has the same remainder as itself.
- Symmetry. If a has the same remainder as b, then b has the same remainder as a. Again, note that since remainders are unique we do not have to worry about ambiguity such as a number having two different remainders.
- Transitivity. If a and b have the same remainder, and b and c have the same remainder, then a and c have the same remainder. Here we are using the uniqueness of the remainder in a crucial way.

(c) For any m the equivalence classes consist of all numbers having the same remainder. According to the Division with Remainder Theorem, the possible remainders are $0, 1, \ldots, m-1$, so these define the equivalence classes. In particular,

- For $m = 3$, the equivalence classes are

$$[1] = \{1, 4, 7, 10, 13, \ldots\};$$
$$[2] = \{2, 5, 8, 11, 14, \ldots\};$$
$$[3] = \{3, 6, 9, 12, 15, \ldots\}.$$

- For $m = 5$, the equivalence classes are

$$[1] = \{1, 6, 11, 16, 21, \ldots\};$$
$$[2] = \{2, 7, 12, 17, 22, \ldots\};$$
$$[3] = \{3, 8, 13, 18, 23, \ldots\};$$
$$[4] = \{4, 9, 14, 19, 24, \ldots\};$$
$$[5] = \{5, 10, 15, 20, 25, \ldots\}.$$

- For $m = 1$, there is a single equivalence class which consists of all the natural numbers:

$$[1] = \{1, 2, 3, 4, 5, \ldots\}.$$

Indeed, this is just a (very roundabout) way of saying that every natural number is divisible by 1.

(d) We can prove that D is reflexive, symmetric, and transitive. However, we can also note that a and b end in the same digit if and only if they have the same remainder when divided by 10. Therefore, D is just M with $m = 10$, which we already know is an equivalence relation.

(Exercise on page 116.)

Solution for Exercise 103 (Advanced Mathematics).

(a) We prove that R is reflexive, symmetric, and transitive.

- Let $x \in \mathbb{R}$ be arbitrary. Then $x - x = 0 \in \mathbb{Z}$, so that xRx. This proves that R is reflexive.
- Let $x, y \in \mathbb{R}$ be arbitrary and suppose xRy. By the definition of R, this means that $y - x \in \mathbb{Z}$. Then $x - y = -(y - x) \in \mathbb{Z}$ so that yRx. This proves that $\mathbb{R}$ is symmetric.
- Let $x, y, z \in \mathbb{R}$ be arbitrary and suppose xRy and yRz. By the definition of R, this means that $y - x \in \mathbb{Z}$ and $z - y \in \mathbb{Z}$. Then $z - x = (z - y) + (y - x) \in \mathbb{Z}$, so that xRz. This proves that R is transitive.

Since R is reflexive, symmetric, and transitive, it is an equivalence relation.

Each real number $r \in [0, 1)$ is a representative of a unique equivalence class:

$$[r] = \{r + m \,:\, m \in \mathbb{Z}\}.$$

None of the real numbers in $[0, 1)$ are related to each other, but each real numbers in $[1, 2)$ is related to exactly one real number in $[0, 1)$, starting with $0R1$ and continuing "in the same order".

Helpful Tip!

Because of the nature of the equivalence classes, mathematicians think of R as "wrapping the real number line around itself"—it goes from 0 to 1 and then "ends up back at 0" when it reaches 1.

This is an algebraic realization of the geometric circle!

(b) The proof is completely analogous to the one above. Reflexivity of Q follows from the fact that $0 \in \mathbb{Q}$; symmetry follows from the fact that if $q \in \mathbb{Q}$ then $-q \in \mathbb{Q}$; and transitivity follows from the fact that if $q, q' \in \mathbb{Q}$ then so is $q + q'$.

The equivalence classes are very difficult to describe, there are as many equivalence classes as there are real numbers (uncountably many)!

There is one (countably infinite) equivalence class for all the rational numbers $\mathbb{Q}$. Other than that, each equivalence class is (countably) infinite and consists of only irrational numbers. If r is an irrational number, then

$$[r] = \{r + q \,:\, q \in \mathbb{Q}\}.$$

Helpful Tip!

This equivalence relation is an important example in the mathematical theory of probability. It is used to prove that there is no straightforward method of assigning length (or probability) to any subset of $\mathbb{R}$.

(Exercise on page 117.)

Solution for Exercise 104 (Counting)**.**

(a) A **partition** of a set A is a collection Ω of *nonempty* subsets of A that are *pairwise disjoint* and *cover* A.

We "unpack" this definition by comparing it to the definition from the reference text (Definition 7.51 on p. 92):

Definition. A collection Ω of subsets of a set A is said to be a **partition** of A if the elements of Ω satisfy:

- The subsets are *nonempty*: for all $X \in \Omega$ we have $X \neq \emptyset$.
- The subsets are *pairwise disjoint*: for all $X, Y \in \Omega$, if $X \neq Y$ then $X \cap Y = \emptyset$.
- The subsets *cover* the set: $\bigcup_{X \in \Omega} X = A$.

As long as your answer included all three of these properties, you were definitely on the right track!

(b) In a pie chart we divide a circular region (the "pie") into a finite number of sectors (the "slices"). For instance,

Each slice of the pie corresponds to a *block* of the partition. Indeed:

- Each slice is nonempty.
- Two slices do not overlap (they are pairwise disjoint).
- The collection of all the slices equal the whole pie (they cover the set).

In mathematics we deal with infinite sets and abstracting from this idea we consider infinitely fine slices. These are very useful generalization of the very basic idea of dividing a pie.

(c) We know that a partition is a collection of subsets. The only subset of $A = \emptyset$ is $\emptyset$. Therefore, the only possibility for Ω is $\{\emptyset\}$. However, a partition must be a collection of *nonempty* subsets. Since there is no such collection, there are *no partitions* of Ω. The number of different partitions of $\emptyset$ is 0.

(d) We know that a partition is a collection of *nonempty subsets*. The nonempty subsets of $\{1\}$ are $\{1\}$, so the only possible partitions of $\{1\}$ is $\Omega = \{\{1\}\}$. Since this is indeed a partition, the number of different partitions of $\{1\}$ is 1.

Similarly, the nonempty subsets of $\{1,2\}$ are $\{1\},\{2\},\{1,2\}$. Suppose Ω is a partition of $\{1,2\}$ which includes $\{1,2\}$. Since the subsets of the partition cannot overlap (the subsets are pairwise disjoint), we cannot include any other subsets, so that $\Omega = \{\{1,2\}\}$ is one possible partition, and it is indeed a partition.

If Ω' is a partition which includes $\{1\}$, then we must also include $\{2\}$, because the subsets of Ω' must cover $\{1,2\}$ and cannot overlap. Similarly, any partition which includes $\{2\}$ must also include $\{1\}$. Therefore, $\Omega' = \{\{1\},\{2\}\}$ is another possible partition, and it is indeed a partition.

Any partition of $\{1,2\}$ includes one of the sets $\{1\},\{2\},\{1,2\}$, so we have checked all the options. In conclusion, $\{1,2\}$ has 2 different partitions.

(e) The nonempty subsets of $\{1,2,3\}$ are

$$\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}.$$

The possible partitions are

$$\Omega_1 = \{\{1\},\{2\},\{3\}\}; \quad \Omega_2 = \{\{1\},\{2,3\}\}; \quad \Omega_3 = \{\{1,2\},\{3\}\};$$
$$\Omega_4 = \{\{1,3\},\{2\}\}; \quad \Omega_5 = \{\{1,2,3\}\}.$$

Thus there are 5 different partitions of $\{1,2,3\}$.

Helpful Tip!

The number of possible partitions of $\{1,2,\ldots,n\}$ is known as the *Bell number* B_n, which also counts the number of possible equivalence relations on $\{1,2,\ldots,n\}$. The first few values are

$$1, 2, 5, 15, 52, 203, 877, \ldots$$

(Exercise on page 119.)

Solution for Exercise 105 (Find the Partitions)**.**

(a) The set $A_1 = \{1, 2, 3, 4, 5, 6\}$.

(i) $\Omega_1 = \{\{1, 2\}, \{2, 3, 4\}, \{4, 5, 6\}\}$ is not a partition of A_1. The sets in Ω_1 are not pairwise disjoint. Indeed, $\{1, 2\} \cap \{2, 3, 4\} = \{2\} \neq \emptyset$.

(ii) $\Omega_2 = \{\{1\}, \{2, 3, 6\}, \{4\}, \{5\}\}$ is a partition of A_1.

(iii) $\Omega_3 = \{\{2, 4, 6\}, \{1, 3, 5\}\}$ is a partition of A_1.

(iv) $\Omega_4 = \{\{1, 4, 5\}, \{2, 6\}\}$ is not a partition of A_1. The sets in Ω_4 do not cover A_1. Indeed, there is no set S in Ω_4 which has 3 as an element (even though $3 \in A_1$).

(v) $\Omega_5 = \{\{1, 2, 3, 4\}, \{5, 6\}, \{\}\}$ is not a partition of A_1. Each set of a partition must be nonempty, whereas $\emptyset \in \Omega_5$.

Note that this is the only condition which fails in this example: if we remove $\emptyset$ to create $\Omega_5' = \{\{1, 2, 3, 4\}, \{5, 6\}\}$ the result is a partition of A_1.

(vi) $\Omega_6 = \{\{1, 2, 3, 4, 5, 6\}\}$ is a partition of A_1.

(b) The set $A_2 = \mathbb{Z}$.

(i) Ω_7 is a partition of $\mathbb{Z}$.

- There are two sets in Ω_7, each of them nonempty.
- An integer cannot be both odd and even, so the two sets are disjoint.
- Every integers is either odd or even, so the two sets cover $\mathbb{Z}$

(ii) Ω_8 is not a partition of $\mathbb{Z}$. There are two sets in Ω_8 and neither of them contains the integer $0 \in \mathbb{Z}$. Therefore, the sets fail to cover $\mathbb{Z}$.

(iii) Ω_9 is a partition of $\mathbb{Z}$. There are three sets in Ω_9:

$$\begin{aligned}
X_1 &= \{n \in \mathbb{Z} : n < -100\}, \\
X_2 &= \{n \in \mathbb{Z} : |n| \leq 100\} = \{n \in \mathbb{Z} : -100 \leq n \leq 100\}, \\
X_3 &= \{n \in \mathbb{Z} : n > 100\}.
\end{aligned}$$

- Each of X_1, X_2, X_3 is nonempty.
- The three sets are pairwise disjoint. This is clear from their definition above, but if we wanted to prove, for example, that X_1, X_2 are disjoint we could argue as follows: if $n < -100$ then $|n| = -n > 100$ so that $n \notin X_2$. Conversely, if $m \in X_2$ then $m \geq -100$ so that $m \notin X_1$.
- The three sets cover $\mathbb{Z}$. Every integer n satisfies $n \leq 100$ or $n > 100$. If the latter, than $n \in X_3$. If the former we use the same reasoning to check whether $n < -100$ or $n \geq -100$. If the former, then $n \in X_1$, and if the latter than $n \in X_2$ (since we now have $-100 \leq n \leq 100$).

(iv) Ω_{10} is not a partition. The sets in $\Omega_1 0$ are not pairwise disjoint. For example, the integer $4 \in \mathbb{Z}$ is an element of both "the set of integers not divisible by 3" and "the set of even integers".

(Exercise on page 120.)

Solution for Exercise 106 (Find the Partitions II)**.**

(a) The set $A_1 = \mathbb{Z} \times \mathbb{Z}$.

(i) Ω_1 is not a partition of A_1. The sets in Ω_1 are not pairwise disjoint. For example, $(1, 2)$ is an element of "the set of pairs (x, y) where x or y is odd" and of "the set of pairs (x, y) where y is even".

(ii) Ω_2 is a partition of A_1.

- Each of the sets in Ω_2 is nonempty.
- Given an ordered pair of integers (x, y) either none, one, or both of x, y are odd and these options are mutually exclusive—so the three sets in Ω_2 do not overlap (are pairwise disjoint).
- For any ordered pair of integers (x, y) either none, one, or both of x, y are odd. Therefore, the sets in Ω_2 cover $\mathbb{Z} \times \mathbb{Z}$.

(iii) Ω_3 is not a partition of A_1. The sets in Ω_3 do not cover A_1. Indeed, the ordered pair $(0, 0) \in \mathbb{Z} \times \mathbb{Z}$ is not an element of any of the sets in Ω_3. This is because the first coordinate x is not positive, the second coordinate y is not positive, and it is not the case that both coordinates are negative.

(iv) Ω_4 is not a partition of A_1. The sets in Ω_4 do not cover A_1. Indeed, $(5, -5) \in \mathbb{Z} \times \mathbb{Z}$ has first coordinate > 0 but second coordinate < 0 and so is not an element of any of the sets in Ω_4.

(v) Ω_5 is not a partition of A_1. The sets in Ω_5 do not cover A_1. Indeed, the ordered pair $(0, 0) \in \mathbb{Z} \times \mathbb{Z}$ has both coordinates 0 and so is not an element of any of the sets in Ω_5.

(b) The set $A_2 = \mathbb{R}$.

(i) Ω_6 is a partition of $\mathbb{R}$.

- None of the three sets comprising Ω_6 is empty.
- The three sets are pairwise disjoint (for example, a real number cannot be both positive and negative).
- The three sets cover $\mathbb{R}$ since any real number is zero, positive, or negative.

(ii) Ω_7 is a partition of $\mathbb{R}$.

- Neither set comprising Ω_7 is empty.
- A real number cannot be both rational and irrational, so the two sets are disjoint.
- Every real number is either rational or irrational, so the two sets cover $\mathbb{R}$.

(iii) Ω_8 is not a partition of $\mathbb{R}$. The sets in Ω_8 are not pairwise disjoint. For example, $[0, 1] \cap [1, 2] = \{1\} \neq \emptyset$.

(iv) Ω_9 is not a partition of $\mathbb{R}$. The sets in Ω_9 do not cover $\mathbb{R}$. For example, there is no interval of the form $(k, k + 1)$ (with $k \in \mathbb{Z}$) that contain the real number 0 as an element.

(v) Ω_{10} is a partition of $\mathbb{R}$.

- Each interval in Ω_{10} is nonempty.
- We prove that the intervals are pairwise disjoint. Let $(j, j+1]$ and $(k, k+1]$ be two distinct intervals, so that $j \neq k$. Assume without loss of generality that $j < k$ (otherwise, swap the names of these two integers). Then $j + 1 \leq k$ (because j, k are integers). Therefore, for any $x \in (j, j + 1]$ we have $x \leq j + 1 \leq k$ so that $x \not> k$ and therefore $x \notin (k, k + 1]$. This proves that $(j, j + 1] \cap (k, k + 1] = \emptyset$.

- We rely on our knowledge of the real numbers to argue that the intervals in Ω_{10} cover $\mathbb{R}$. Below we argue this slightly more carefully, but it's acceptable to just state this for the purpose of this question.

  It is clear that the integer k is an element of the interval $(k-1, k]$. We now show that each non-integer real number is an element of one of the intervals. We need the familiar fact from school that every real number can be written as a decimal expansion:

  $$\pm d_n d_{n-1} \ldots d_1 . e_1 e_2 e_3 \ldots$$

  For example, $\pi = 3.141519\ldots$ and $-e = -2.7182818\ldots$. For any non-integer real number r, let r' be the part of r "before the decimal point" (in the notation above: $d_n d_{n-1} \ldots d_1$ or $-d_n d_{n-1} \ldots d_1$). Then for any non-integer real number r we have
  - if $r > 0$ then $r \in (r', r'+1]$.
  - if $r < 0$ then $r \in (r'-1, r']$.

(Exercise on page 121.)

Solution for Exercise 107 (Constructing Partitions)**.**

Note that there are (infinitely!) many correct solutions to this exercise. Make sure your partitions include sets that are nonempty, pairwise disjoint, and cover all of $\mathbb{N}$.

(a) For example, $\Omega = \{\{1\}, \{2\}, \{n \in \mathbb{N} : n > 2\}\}$. Another example is

$$\Omega' = \{\{1, 2, \dots, 10\}, \{11, 12, \dots, 99\}, \{n \in \mathbb{N} : n \geq 100\}\}.$$

(b) For example, $\Omega = \{\{1\}, \{2\}, \{3\}, \dots\}$. Another example is

$$\Omega' = \{\{1\}, \{2, 3\}, \{4, 5, 6\}, \{7, 8, 9, 10\}, \dots\}.$$

(c) For example, Ω consisting of the three sets

$$\begin{aligned}
&\{\{n \in \mathbb{N} : n = 3k, \text{ for some natural number } k\} && = \{3, 6, 9, 12, \dots\}; \\
&\{n \in \mathbb{N} : n = 3k + 1, \text{ for some natural number } k\} && = \{1, 4, 7, 10, \dots\}; \\
&\{n \in \mathbb{N} : n = 3k + 2, \text{ for some natural number } k\} && = \{2, 5, 8, 11, \dots\}.
\end{aligned}$$

Another example is Ω' consisting of the three sets

$$\begin{aligned}
&\text{the even natural numbers} && = \{2, 4, 6, 8, \dots\}; \\
&\text{the odd natural numbers that are not divisible by 5} && = \{1, 3, 7, 9, 11, 13, 17, 19, \dots\}; \\
&\text{the odd natural numbers that are divisible by 5} && = \{5, 15, 25, 35, 45, \dots\}.
\end{aligned}$$

(d) A partition of $\mathbb{N}$ is a collection of subsets of $\mathbb{N}$ that are nonempty, pairwise disjoint, and cover $\mathbb{N}$.

- $\{\emptyset, \mathbb{N}\}$ is a non-example of a partition. The sets are pairwise disjoint and cover $\mathbb{N}$, but not all of them are nonempty.

  Adding $\emptyset$ to any of the partitions from parts (a)–(c) will result other such non-examples.

- $\{\{1, 2\}, \{2, 3\}, \{3, 4\}, \dots\}$ is a non-example of a partition. The sets are nonempty and cover $\mathbb{N}$ but are not pairwise disjoint.

  Another non-example is the collection consisting of the two sets:

  $$\{1\} \cup \{2, 4, 6, 8, \dots\}; \qquad \{1, 3, 5, 7, \dots\}.$$

- $\{\{2\}, \{3\}, \{4\}, \{5\}, \dots\}$ is a non-example of a partition. The sets are nonempty and pairwise disjoint, but fail to cover $\mathbb{N}$.

  Another non-example is the collection consisting of the two sets

  $$\{1, 4, 7, 10, \dots\}; \qquad \{2, 5, 8, 11, \dots\}.$$

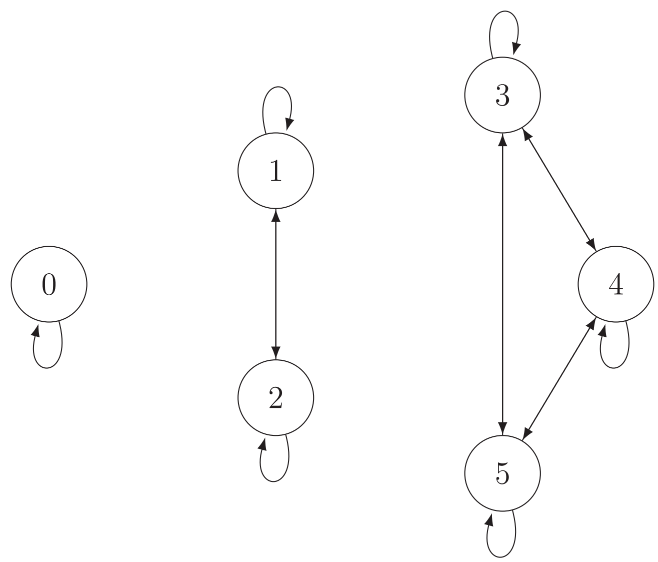
(Exercise on page 122.)

Solution for Exercise 108 (Relations from subsets).

(a) $\Omega_1 = \{\{0\}, \{1, 2\}, \{3, 4, 5\}\}$. The relation R_{Ω_1} is given by the ordered pairs

$$R_{\Omega_1} = \{(0,0), (1,1), (1,2), (2,1), (3,3), (3,4), (3,5), (4,4), (4,3), (4,5), (5,3), (5,4), (5,5)\}.$$

The corresponding digraph is:

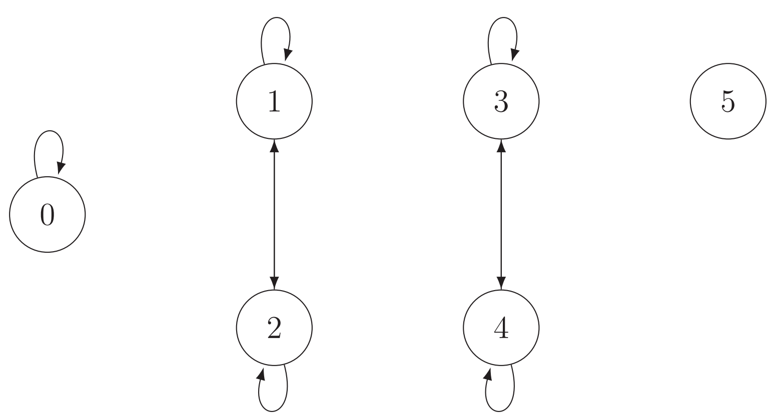


R_{Ω_1} is an equivalence relation on A and Ω_1 is a partition of A. Moreover, the equivalence classes of R_{Ω_1} are precisely the blocks of the partition Ω_1.

(b) $\Omega_2 = \{\{0\}, \{1, 2\}, \{3, 4\}\}$. The relation R_{Ω_2} is given by the ordered pairs

$$R_{\Omega_2} = \{(0,0), (1,1), (1,2), (2,1), (3,3), (3,4), (4,4), (4,3), \}.$$

The corresponding digraph is:

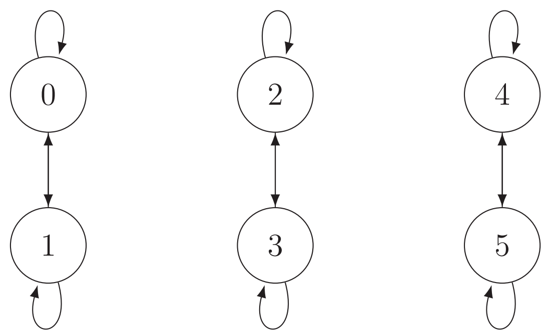


R_{Ω_2} is not an equivalence relation; reflexivity fails since 5 is not related to 5. Similarly, Ω_2 is not a partition because it fails to cover A, no set in Ω has 5 as an element.

(c) $\Omega_3 = \{\{0,1\},\{2,3\},\{4,5\}\}$. The relation R_{Ω_3} is given by the ordered pairs

$$R_{\Omega_3} = \{(0,0),(0,1),(1,0),(1,1),(2,2),(2,3),(3,2),(3,3),(4,4),(4,5),(5,4),(5,5)\}.$$

The corresponding digraph is:

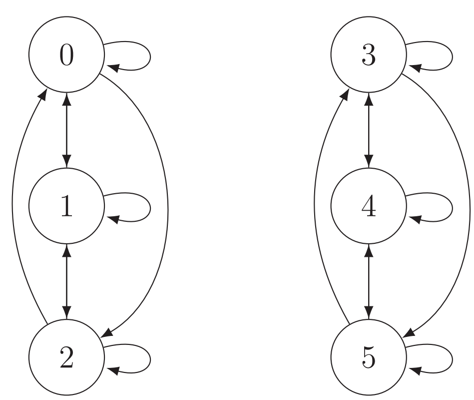


R_{Ω_3} is an equivalence relation on A and Ω_3 is a partition of A. Moreover, the equivalence classes of R_{Ω_3} are precisely the blocks of the partition Ω_3.

(d) $\Omega_4 = \{\{0,1,2\},\{3,4,5\}\}$. The relation R_{Ω_4} is given by the ordered pairs

$$R_{\Omega_4} = \{(0,0),(0,1),(0,2),(1,0),(1,1),(1,2),(2,0),(2,1),(2,2),(3,3),(3,4),(3,5),$$
$$(4,3),(4,4),(4,5),(5,3),(5,4),(5,5)\}.$$

The corresponding digraph is:

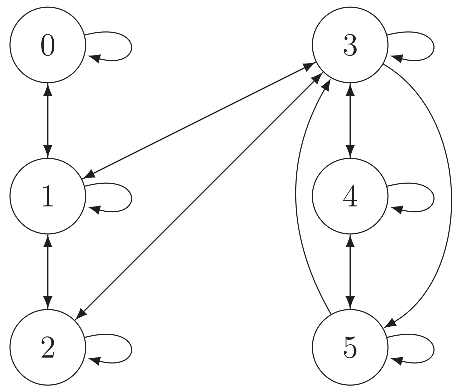


R_{Ω_4} is an equivalence relation on A and Ω_4 is a partition of A. Moreover, the equivalence classes of R_{Ω_4} are precisely the blocks of the partition Ω_4.

(e) $\Omega_5 = \{\{0,1\},\{1,2,3\},\{3,4,5\}\}$. The relation R_{Ω_5} is given by the ordered pairs

$$R_{\Omega_4} = \{(0,0),(0,1),(1,0),(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3),(3,4),(3,5),$$
$$(4,3),(4,4),(4,5),(5,3),(5,4),(5,5)\}.$$

The corresponding digraph is:

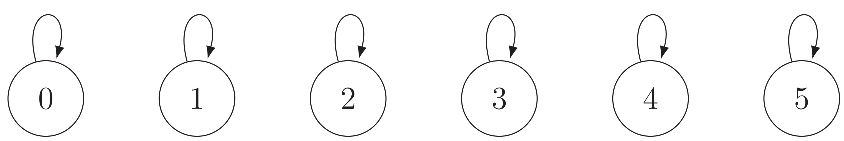


R_{Ω_5} is not an equivalence relation, since it is not transitive. Indeed, $(0,1) \in R_{\Omega_5}$ and $(1,2) \in R_{\Omega_5}$, but $(0,2) \notin R_{\Omega_5}$. Similarly, Ω_5 is not a partition because the sets in Ω_5 are not pairwise disjoint. Indeed, $\{0,1\} \cap \{1,2,3\} = \{1\} \neq \emptyset$.

(f) $\Omega_6 = \{\{0\},\{1\},\{2\},\{3\},\{4\},\{5\}\}$. The relation R_{Ω_6} is given by the ordered pairs

$$R_{\Omega_6} = \{(0,0),(1,1),(2,2),(3,3),(4,4),(5,5)\}.$$

The corresponding digraph is:



R_{Ω_6} is an equivalence relation on A and Ω_6 is a partition of A. Moreover, the equivalence classes of R_{Ω_6} are precisely the blocks of the partition Ω_6.

(Exercise on page 123.)

Solution for Exercise 109 (Relations and Partitions)**.**

(a) Suppose $a, b \in A$ are such that $aR_\Omega b$. Then there exists some $X \in \Omega$ such that $a, b \in X$. In that case, for the very same X we have $b, a \in X$ so that $bR_\Omega a$. Since $a, b \in A$ were arbitrary, this proves that R_Ω is symmetric.

(b) Note that Ω covers A if and only if for every $a \in A$ there exists some $X \in A$ such that $a \in X$ (by the definition of cover, or of $\bigcup_{X\in\Omega} X$); if and only if for every $a \in A$ we have $aR_\Omega a$ (by the definition of the relation associated to a collection of sets); if and only if R_Ω is reflexive (by the definition of reflexivity).

(c) Suppose the sets in Ω are pairwise disjoint. Let $a, b, c \in A$ be arbitrary and suppose $aR_\Omega b$ and $bR_\Omega c$. From $aR_\Omega b$ we conclude there is some $X \in \Omega$ such that $a, b \in X$. From $bR_\Omega c$ we conclude there is some $Y \in \Omega$ such that $b, c \in Y$. Since the sets in Ω are pairwise disjoint and $b \in X \cap Y$, we must have $X = Y$. Therefore, $a, c \in X$ so that $aR_\Omega c$. This proves that R_Ω is transitive.

(d) A simple example where the sets in Ω are not pairwise disjoint but R_Ω is transitive is $A = \{1, 2\}$ and $\Omega = \{\{1\}, \{2\}, \{1, 2\}\}$. In general, including A in Ω results in a trivial transitive relation because every element is related to every other element.

(e) We know that R_Ω is symmetric. If Ω is a partition, then it covers A, so that R_Ω is reflexive. Moreover, if Ω is a partition, its sets are pairwise disjoint, so that R_Ω is transitive. Therefore, if Ω is a partition, R_Ω is reflexive, symmetric, and transitive, and hence an equivalence relation on A.

(f) We claim that the equivalence classes of R_Ω are precisely the blocks in Ω.

Since Ω is a partition, it covers A. Therefore, for any $a \in A$ there is at least one $X \in \Omega$ such that $a \in X$.

Since Ω is a partition, its sets are pairwise disjoint, so for any $a \in A$ there is at most one $X \in \Omega$ such that $a \in X$.

Taking together, these two facts show that for any $a \in A$ there is a unique $X \in \Omega$ such that $a \in X$, let's call it X_a for clarity.

We claim that for any $a \in A$ its equivalence class $[a]$ is precisely X_a. Indeed, $b \in [a]$ if and only if $aR_\Omega b$ (by the definition of an equivalence class) if and only if there is some $Y \in \Omega$ such that $a, b \in Y$ (by the definition of R_Ω). However, there is a unique set in Ω containing a, so that $a, b \in Y$ if and only if ($Y = X_a$ and) $a, b \in X_a$. This proves that $[a] = X_a$.

(g) Suppose R_Ω is an equivalence relation on A. Let Ω be the set of equivalence classes of R_Ω. We prove that Ω is a partition of A.

- The sets in Ω are nonempty. Indeed, for any $a \in A$, the equivalence class $[a]$ is nonempty; below we show that $a \in [a]$.
- Ω covers A. Let $a \in A$ be arbitrary. Since R_Ω is an equivalence relation, it is reflexive, so that $aR_\Omega a$. This proves that $a \in [a]$. In particular, there is some $X \in \Omega$ such that $a \in X$ (namely, $X = [a]$). This proves that $\bigcup_{X\in\Omega} X = A$, i.e. that Ω covers A.
- The sets in Ω are pairwise disjoint. Suppose $[a] \neq [b]$, we prove that $[a] \cap [b] = \emptyset$. Indeed, suppose towards contradiction that $c \in [a] \cap [b]$, we prove that $[a] = [b]$, contradicting the assumption that $[a] \neq [b]$.
  We prove that $[a] \subseteq [b]$, the proof that $[b] \subseteq [a]$ is completely analogous (interchange the roles of a and b in the proof). To prove that $[a] \subseteq [b]$ we need to show that $x \in [a]$ implies $x \in [b]$. That is, that $aR_\Omega x$ implies $bR_\Omega x$. The idea is to use transitivity via c. Below are the details.

Let $x \in [a]$ be arbitrary, so that $aR_\Omega x$. Since R_Ω is an equivalence relation, it is symmetric, so that $xR_\Omega a$.

Since $c \in [a]$ we have $aR_\Omega c$. Since R_Ω is an equivalence relation, it is transitive, so that from $xR_\Omega a$ and $aR_\Omega c$ we may conclude $xR_\Omega c$.

Since $c \in [b]$ we have $bR_\Omega c$ and by symmetry $cR_\Omega b$. From $xR_\Omega c$ and $cR_\Omega b$ we have by transitivity $xR_\Omega b$. By symmetry $bR_\Omega x$ so that $x \in [b]$. This proves that $[a] \subseteq [b]$.

(Exercise on page 124.)

Solution for Exercise 110 (Relations and Partitions II)**.**

Suppose R is an equivalence relation on A. Let Ω be the set of equivalence classes of R. We prove that Ω is a partition of A.

- The sets in Ω are nonempty. Indeed, for any $a \in A$, the equivalence class $[a]$ is nonempty; below we show that $a \in [a]$.

- Ω covers A. Let $a \in A$ be arbitrary. Since R is an equivalence relation, it is reflexive, so that aRa. This proves that $a \in [a]$. In particular, there is some $X \in \Omega$ such that $a \in X$ (namely, $X = [a]$). This proves that $\bigcup_{X\in\Omega} X = A$, i.e. that Ω covers A.

- The sets in Ω are pairwise disjoint. Suppose $[a] \neq [b]$, we prove that $[a] \cap [b] = \emptyset$. Indeed, suppose towards contradiction that $c \in [a] \cap [b]$, we prove that $[a] = [b]$, contradicting the assumption that $[a] \neq [b]$.

  We prove that $[a] \subseteq [b]$, the proof that $[b] \subseteq [a]$ is completely analogous (interchange the roles of a and b in the proof). To prove that $[a] \subseteq [b]$ we need to show that $x \in [a]$ implies $x \in [b]$. That is, that aRx implies bRx. The idea is to use transitivity via c. Below are the details.

  Let $x \in [a]$ be arbitrary, so that aRx. Since R is an equivalence relation, it is symmetric, so that xRa.

  Since $c \in [a]$ we have aRc. Since R is an equivalence relation, it is transitive, so that from xRa and aRc we may conclude xRc.

  Since $c \in [b]$ we have bRc and by symmetry cRb. From xRc and cRb we have by transitivity xRb. By symmetry bRx so that $x \in [b]$. This proves that $[a] \subseteq [b]$.

(Exercise on page 125.)

Solution for Exercise 111 (Refinements)**.**

(a) Yes, Ω_1 is a refinement of Ω_2. We start by noting that Ω_1, Ω_2 are indeed partitions of A, so the question makes sense. Next, we note that every set in Ω_1 is a subset of a set in Ω_2. Indeed,

$$\{1,2\} \subseteq \{1,2,3\}, \qquad \{3\} \subseteq \{1,2,3\}, \qquad \{4\} \subseteq \{4,5,6\}, \qquad \{5,6\} \subseteq \{4,5,6\}.$$

Therefore, Ω_1 is a refinement of Ω_2,

(b) There are many correct answers. The important point is to partition one or more of the sets in Ω_2. For example, we can take $\{1,2,3\}$ and partition it into $\{1\}$ and $\{2,3\}$ to obtain

$$\Omega_1 = \{\{1\},\{2,3\},\{4,5,6\}\}.$$

Note that Ω_1 is a partition of A and every block in Ω_1 is a subset of a block in Ω_2.

(c) First, let's make sure Ω_1 is a partition of A. Its elements are nonempty, disjoint, and their union is equal to A, so Ω_1 is a partition of A. Next, we check if every set in Ω_1 is a subset of a set in Ω_2. We have

$$\{1,2\} \subseteq \{1,2,3\}, \{3\} \subseteq \{1,2,3\},$$
$$\{4\} \subseteq \{4,5,6\}, \text{ and } \{5,6\} \subseteq \{4,5,6\}.$$

Therefore, Ω_1 is a refinement of Ω_2,

(d) The claim is true, P_1 is a refinement of P_3. To prove this, let $X \in P_1$ be arbitrary. Since P_1 is a refinement of P_2 there exists some $Y \in P_2$ such that $X \subseteq Y$.

Since P_2 is a refinement of P_3, there exists some $Z \in P_3$ such that $Y \subseteq Z$. Then $X \subseteq Y \subseteq Z$ and by the transitivity of inclusion we conclude $X \subseteq Z$.

Since X was an arbitrary block of P_1, this proves that $\forall X \in P_1 \, \exists Z \in P_3 \, (X \subseteq Z)$, i.e. that P_1 is a refinement of P_3.

(e) Suppose Q_1, Q_2 are partitions of C and that Q_1 is a refinement of Q_2. Let R_{Q_1}, R_{Q_2} be the equivalence relations corresponding to Q_1, Q_2, respectively. We claim that $R_{Q_1} \subseteq R_{Q_2}$, i.e. for any $x, y \in C$ we have $(xR_{Q_1}y) \implies (xR_{Q_2}y)$.

Indeed, let $x, y \in C$ be arbitrary and suppose $xR_{Q_1}y$. By the definition of R_{Q_1} this means there exists some $X \in Q_1$ such that $x, y \in X$. Since Q_1 is a refinement of Q_2, there exists some $Y \in Q_2$ such that $x, y \in Y$. By the definition of R_{Q_2}, this means that $xR_{Q_2}y$. This concludes the proof.

Note that in general, $xR_{Q_2}y$ does *not* imply $xR_{Q_1}y$, so that R_{Q_1} is "more discerning" or "makes finer distinctions" (fewer elements are equivalent), that's why Q_1 is called a *refinement* of Q_2. Intuitively, the blocks in Ω_1 are "smaller" or "more fine grained" than those in Ω_2.

(Exercise on page 126.)

Solution for Exercise 112 (Representatives)**.**

We prove that $(a) \implies (b) \implies (c) \implies (a)$.

- Suppose $[a] = [b]$. Since R is reflexive, we have aRa, so that $a \in [a]$ (by the definition of equivalence class). Since $[a] = [b]$, we conclude $a \in [b]$.
- Suppose $a \in [b]$. By the definition of equivalence class, this means aRb.
- Suppose aRb. We first prove that $[a] \subseteq [b]$. Let $x \in [a]$ be arbitrary. By the definition of equivalence class, xRa. Since R is an equivalence relation, it is transitive, so from xRa and aRb we conclude xRb. By the definition of equivalence class, this shows that $x \in [b]$. Since $x \in [a]$ was arbitrary, this proves that $[a] \subseteq [b]$.

  Since R is an equivalence relation, it is symmetric, so from aRb, we conclude bRa. By the proof from the previous paragraph, this implies $[b] \subseteq [a]$.

  Since $[a] \subseteq [b]$ and $[b] \subseteq [a]$, we conclude $[a] = [b]$.

(Exercise on page 127.)

Solution for Exercise 113 (Operations)**.**

(a) We prove that $\oplus_1$ is well-defined. Towards that end, suppose $[a] = [a']$ and $[b] = [b']$. We must show that

$$[a] \oplus_1 [b] = [a'] \oplus_1 [b'].$$

By definition of $\oplus_1$ we have

$$[a] \oplus_1 [b] = [a \boxplus_1 b] = [a]$$

where the last equality follows from the definition of $\boxplus_1$. Similarly,

$$[a'] \oplus_1 [b'] = [a' \boxplus_1 b'] = [a'].$$

Since $[a] = [a']$ by assumption, we conclude that $[a] \oplus_1 [b] = [a'] \oplus_1 [b']$, as we wanted to show.

(b) We prove that $\oplus_2$ is well-defined. Towards that end, suppose $[a] = [a']$ and $[b] = [b']$. We must show that

$$[a] \oplus_2 [b] = [a'] \oplus_2 [b'].$$

The left side is $[a \boxplus_2 b]$ and the right side is $[a' \boxplus_2 b]$. Since $x \boxplus_2 y$ is 0 or 1 according to the parity of $x + y$, to prove that $[a \boxplus_2 b] = [a' \boxplus_2 b']$, we must show that $a + b$ and $a' + b'$ have the same parity. That is, we must show that $2|((a' + b') - (a + b))$.

Since $[a] = [a']$ and $[b] = [b']$, (by Exercise 112) we have $a \equiv_{10} a'$ and $b \equiv_{10} b'$. By the definition of $\equiv_{10}$, this means that $10|(a' - a)$ and $10|(b' - b)$. It follows that

$$10|(a' - a) + (b' - b) = (a' + b' - (a + b)).$$

Indeed, let $r, s \in \mathbb{Z}$ such that $(a' - a) = 10r$ and $(b' - b) = 10s$. Then, $(a' - a) + (b' - b) = 10(r + s)$.

Since $10|(a' + b' - (a + b))$, we have $2|(a' + b' - (a + b))$ (writing $(a' + b' - (a + b))$ as $10k$ we see that $10k = 2(5k)$). Thus, $a' + b'$ and $a + b$ have the same parity.

(c) We prove that $\oplus_3$ is *not* well-defined. Indeed, $[0] = [10]$ and $[1] = [11]$, but

$$[0] \oplus_3 [1] = [0 \boxplus_3 1] = [1]$$

whereas

$$[10] \oplus_3 [11] = [10 \boxplus_3 11] = [0]$$

and $[1] \neq [0]$.

(d) We prove that $\oplus_4$ is *not* well-defined. Indeed, $[0] = [10]$ and $[1] = [1]$, but

$$[0] \oplus_4 [1] = [0 \boxplus_4 1] = [0]$$

whereas

$$[10] \oplus_4 [1] = [10 \boxplus_4 1] = [1]$$

and $[0] \neq [1]$.

(e) We prove that $\oplus_5$ is well-defined. Towards that end, suppose $[a] = [a']$ and $[b] = [b']$. We must prove that

$$[a] \oplus_5 [b] = [a'] \oplus_5 [b'].$$

The left side is $[a \boxplus_5 b] = [2a + 3b]$ and the right side is similarly $[2a' + 3b']$. Therefore, we must prove that $10|(2a' + 3b' - (2a + 3b))$.

Since $[a] = [a']$ and $[b] = [b']$, we have $10|(a' - a)$ and $10|(b' - b)$. It follows that $10|(2a' - 2a)$ and $10|(3b' - 3b)$. Therefore, $10|(2a' - 2a) + (3b' - 3b) = (2a' + 3b' - (2a + 3b))$, proving that $[2a + 3b] = [2a' + 3b']$.

(Exercise on page 128.)

Solution for Exercise 114 (Properties)**.**

The proofs of all of these properties proceed in the same manner: since they hold for representatives, they must hold for the equivalence classes too, because the operation on equivalence classes is defined in terms of the operation on the representative. This is what is meant by saying that the properties are *inherited*. In most mathematics text, one wouldn't expand on that, but to help us understand this claim better, we prove the claim in detail.

(a) Suppose $\boxplus$ is associative. Let $A, B, C \in X/R$ be arbitrary equivalence classes with representatives $a, b, c \in X$ (respectively). Then,

$$\begin{aligned}(A \oplus B) \oplus C &= ([a] \oplus [b]) \oplus [c] \\ &= [a \boxplus b] \oplus [c] && \text{by the definition of } \oplus \\ &= [(a \boxplus b) \boxplus c] && \text{by the definition of } \oplus \\ &= [a \boxplus (b \boxplus c)] && \text{since } \boxplus \text{ is associative} \\ &= [a] \oplus [b \boxplus c] && \text{by the definition of } \oplus \\ &= [a] \oplus ([b] \oplus [c]) && \text{by the definition of } \oplus \\ &= A \oplus (B \oplus C).\end{aligned}$$

This proves that $\oplus$ is associative.

(b) Let $A, B \in X/R$ be arbitrary equivalence classes with representatives $a, b \in X$ (respectively). Then,

$$\begin{aligned}A \oplus B &= [a] \oplus [b] \\ &= [a \boxplus b] && \text{by the definition of } \oplus \\ &= [b \boxplus a] && \text{since } \boxplus \text{ is commutative} \\ &= [b] \oplus [a] && \text{by the definition of } \oplus \\ &= B \oplus A.\end{aligned}$$

This proves that $\oplus$ is commutative.

(c) Let $o \in X$ be the identity element of $\boxplus$. We claim that its equivalence class $[o]$ is the identity element of $\oplus$. Indeed, let $A \in X/R$ be an arbitrary equivalence class, with representative $a \in X$. Then,

$$\begin{aligned}[o] \oplus A &= [o] \oplus [a] \\ &= [o \boxplus a] && \text{by the definition of } \oplus \\ &= [a] && \text{since } o \text{ is an identity element for } \boxplus \\ &= A.\end{aligned}$$

This proves that $[o]$ is an identity element for $\oplus$.

(d) Let $A \in X/R$ be an arbitrary equivalence class with representative $a \in X$. Let $b \in X$ be the $\boxplus$ inverse of a. We claim that $[b]$ is an inverse for A. That is, we claim that $A \oplus [b] = [o]$. Indeed,

$$\begin{aligned}A \oplus [b] &= [a] \oplus [b] \\ &= [a \boxplus b] && \text{by the definition of } \oplus \\ &= [o] && \text{since } b \text{ is a } \boxplus\text{-inverse of } a.\end{aligned}$$

This proves that every equivalence class has an $\oplus$-inverse.

(e) Each proof above uses the fact that $\oplus$ is defined implicitly, when we choose representatives for the equivalence classes. Each equivalence class may have many (even infinitely many) different representatives, but we are implicitly claiming that it does not matter which one is chosen. This is precisely what it means for $\oplus$ to be well-defined.

Helpful Tip!

You have just proved that if $(X, \boxplus)$ is a (commutative) group, then so is the set of equivalence classes $(X/R, \oplus)$. Groups are central objects in modern mathematics and physics, the courses MAT 301 and MAT 347 explore group theory in depth.

(Exercise on page 129.)

Solution for Exercise 115 (Integers).

(a) We show that $\sim$ is reflexive, symmetric, and transitive.

- For any $(a, b) \in Z$ note that $a + b = b + a$ so that $(a, b) \sim (a, b)$. This proves that $\sim$ is reflexive.
- Let $(a, b), (c, d) \in Z$ and suppose $(a, b) \sim (c, d)$. Then $a+d = b+c$ and therefore $c+b = d+a$, so that $(c, d) \sim (a, b)$. This proves that $\sim$ is symmetric.
- Let $(a, b), (c, d), (e, f) \in Z$ and suppose $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$. Then $a + d = b + c$ and $c + f = d + e$. Adding the second equality to the first we find

$$a + d + c + f = b + c + d + e$$

and cancelling $c + d$ from both sides we conclude $a + f = b + e$, so that $(a, b) \sim (e, f)$. This proves that $\sim$ is transitive.

(b) We start by proving that the equivalence classes are all distinct. First, no class of the form $[(n, 0)]$ (with $n \in \mathbb{N} \cup \{0\}$) is equivalent to any class of the form $[(0, m)]$ (with $m \in \mathbb{N}$). Indeed, $(n, 0) \sim (0, m)$ if and only if $n + m = 0 + 0$ if and only if $n = m = 0$, but $m \neq 0$.

Next, if $m \neq n$ then $(n, 0) \not\sim (m, 0)$ because $n + 0 = n \neq m = 0 + m$. Similarly, $(0, n) \not\sim (0, m)$ because $0 + m \neq n + 0$. This concludes the proof that the equivalence classes $[(n, 0)]$ and $[(0, n)]$ are all distinct.

Next, we show that these distinct representatives form a complete system: suppose $(a, b) \in Z$ is an arbitrary element.

- If $a \geq b$, then $[(a, b)] = [(a - b, 0)]$. Indeed, $(a, b) \sim (a - b, 0)$ because $a + 0 = b + (a - b)$.
- If $a < b$, then $[(a, b)] = [(0, b - a)]$. Indeed, $(a, b) \sim (0, b - a)$ because $a + (b - a) = b + 0$.

This proves that every $(a, b) \in Z$ belongs to (exactly) one of the equivalence classes $[(n, 0)]$ or $[(0, n)]$.

(c) Suppose $(a, b) \sim (a', b')$ and $(c, d) \sim (c', d')$. In that case, $[(a, b)] = [(a', b')]$ and $[(c, d)] = [(c', d')]$, so for $\boxplus$ to be well-defined, we have to prove that

$$[(a, b)] \boxplus [(c, d)] = [(a', b')] \boxplus [(c', d')].$$

By the definition of $\boxplus$, the left side is $[(a+c, b+d)]$ and the right side is $[(a'+c', b'+d')]$, to prove that these two equivalence classes are the same, we must prove that $(a+c, b+d) \sim (a'+c', b'+d')$.

By the assumption that $[(a, b)] = [(a', b')]$ we have $(a, b) \sim (a', b')$ so that $a + b' = b + a'$. Similarly, from $[(c, d)] = [(c', d')]$ we have $c + d' = d + c'$. Adding these two equalities we have

$$a + c + b' + d' = b + d + a' + c'$$

which (by the definition of $\sim$) means $(a + c, b + d) \sim (a' + c', b' + d')$. This proves that $\boxplus$ is well-defined.

(d) The natural numbers $m, n \in \mathbb{N} \cup \{0\}$ have been identified with $[(m, 0)]$ and $[(n, 0)]$ (respectively), and their sum $m + n$ has been identified with $[(m + n, 0)]$. The operation we have just defined on Z works well with this identification because

$$[(m, 0)] \boxplus [(n, 0)] = [(m + n, 0)]$$

so the Z-sum $\boxplus$ matches the regular $\mathbb{N}$-sum $+$ via our identification.

Helpful Tip!

This is the reason we can think of Z as *extending* $\mathbb{N} \cup \{0\}$, or of $[(n,0)]$ as a *copy of* $\mathbb{N} \cup \{0\}$ inside Z. Any property of $\mathbb{N} \cup \{0\}$ under addition will be preserved by the subset of Z consisting of the equivalence classes $[(n,0)]$ under $\boxplus$. For example, commutativity, associativity, and the fact that 0 is a neutral element. (See Exercise 114.)

(e) Let $n \in \mathbb{N} \cup \{0\}$. Then, n is identified with $[(n,0)]$ and $-n$ with $[(0,n)]$. Then, $n + (-n)$ is by definition

$$[(n,0)] \boxplus [(0,n)] = [(n+0, 0+n)] = [(n,n)].$$

where we have used the definition of the $\boxplus$ operation. Finally, $[(n,n)] = [(0,0)]$ because $(n,n) \sim (0,0)$. Indeed, $n + 0 = n + 0$. This proves that $n - n = 0$.

(f) The numbers 2, 3, and 5 are identified with the equivalence classes $[(2,0)]$, $[(3,0)]$, and $[(5,0)]$, respectively. Similarly, $-2, -3, -5$ are identified with $[(0,2)]$, $[(0,3)]$, $[(0,5)]$, respectively.

The operation $5 - 2$ is by definition $5 + (-2)$ which is

$$[(5,0)] \boxplus [(0,2)] = [(5+0, 0+2)] = [(5,2)].$$

The claim that $5 - 2 = 3$ is therefore the claim that $[(5,2)] = [(3,0)]$ or that $(5,2) \sim (3,0)$. Indeed, $5 + 0 = 2 + 3$. This proves that $5 - 2 = 0$.

The operation $2 - 5$ is by definition $2 + (-5)$ which is

$$[(2,0)] \boxplus [(0,5)] = [(2+0, 0+5)] = [(2,5)].$$

The claim that $2 - 5 = -3$ is therefore the claim that $[(2,5)] = [(0,3)]$ or that $(2,5) \sim (0,3)$. Indeed, $2 + 3 = 5 + 0$. This proves that $2 - 5 = -3$.

(g) $[(a,b)]$ is identified with $n \in \mathbb{N} \cup \{0\}$ if and only if $[(a,b)] = [(n,0)]$, if and only if $(a,b) \sim (n,0)$, if and only if $a + 0 = b + n$.

Now, $a + 0 = b + n$ if and only if $b + n = a + 0$, if and only if $[(b,a)] = [(0,n)]$, if and only if $[(b,a)]$ is identified with $-n$.

(h) Suppose $[(a,b)] = [(a',b')]$. We need to prove that $\boxminus[(a,b)] = \boxminus[(a',b')]$. By the definition of $\boxminus$ the left side is $[(b,a)]$ and the right side is $[(b',a')]$, so we need to prove that $(b,a) \sim (b',a')$. This holds if and only if $b + a' = a + b'$, if and only if $a + b' = b + a'$, if and only if $[(a,b)] = [(a',b')]$.

(i) We have

$$\begin{aligned}[(a,b)] \boxminus [(c,d)] &= [(a,b)] \boxplus (\boxminus[(c,d)]) \\ &= [(a,b)] \boxplus [(d,c)] \\ &= [(a+d, b+c)].\end{aligned}$$

Therefore, we have $e = a + d$ and $f = b + c$. We already know that this operation is well-defined, because each of $\boxplus$ and $\boxminus$ is well-defined.

We do not need to say anything more, but let's think about this last claim more carefully. Suppose $[(a,b)] = [(a',b')]$ and $[(c,d)] = [(c',d')]$. Since $\boxminus$ is well-defined we have $\boxminus[(c,d)] = \boxminus[(c',d')]$. Since $\boxplus$ is well-defined we have

$$[(a,b)] \boxplus (\boxminus[(c,d)]) = [(a',b')] \boxplus (\boxminus[(c',d')])$$

proving that

$$[(a, b)] \boxminus [(c, d)] = [(a', b')] \boxminus [(c', d')].$$

Finally, we find that $2 - 5$ corresponds to

$$[(2, 0)] \boxminus [(5, 0)] = [(2 + 0, 0 + 5)] = [(2, 5)].$$

Now, $(2, 5) \sim (0, 3)$ since $2 + 3 = 0 + 5$, proving that $2 - 5 = -3$.

(Exercise on page 131.)

Solution for Exercise 116 (Rationals)**.**

(a) We prove that $\sim$ is reflexive, symmetric, and transitive

- For any $(a, b) \in Q$ we have $ab = ba$, so that $(a, b) \sim (a, b)$. This proves that $\sim$ is reflexive.
- Let $(a, b), (c, d) \in Q$ be arbitrary, and suppose $(a, b) \sim (c, d)$ so that $ad = bc$. Then $cb = da$ so that $(c, d) \sim (a, b)$. This proves that $\sim$ is symmetric.
- Let $(a, b), (c, d), (e, f) \in Q$ be arbitrary and suppose $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$. Then $ad = bc$ and $cf = de$. Using the first equality and then the second, we therefore have

$$adcf = adde = bcde.$$

Since $(c, d) \in Q$, we know that $d \neq 0$, so we can cancel d from both sides to conclude $acf = bce$. We would also like to cancel c, but this is not necessarily possible, so we need to distinguish between two cases.

  - If $c \neq 0$ we can cancel c to conclude $af = be$ so that $(a, b) \sim (e, f)$.
  - If $c = 0$ then from $(a, b) \sim (c, d)$ we conclude $ad = bc = 0$. Since $(a, b) \in Q$, we know that $b \neq 0$, so we must have $a = 0$. Similarly, from $(c, d) \sim (e, f)$ we conclude $0 = cf = de$. Since $(e, f) \in Q$, we know that $f \neq 0$, so we must have $e = 0$.
    We can now conclude that $(a, b) \sim (e, f)$ since $af = 0 = be$.

We have found that $(a, b) \sim (e, f)$, which proves that $\sim$ is symmetric.

(b) Suppose $[(a, b)] = [(a', b')]$ and $[(c, d)] = [(c', d')]$. To prove that $\otimes$ is well-defined we must prove that

$$[(a, b)] \otimes [(c, d)] = [(a', b')] \otimes [(c', d')].$$

By the definition of $\otimes$, the left side is $[(ac, bd)]$ and the right side is $[(a'c', b'd')]$, so we must show that $(ac, bd) \sim (a'c', b'd')$.

Since $[(a, b)] = [(a', b')]$, we have $(a, b) \sim (a', b')$ so that $ab' = ba'$. Similarly, from $[(c, d)] = [(c', d')]$, we conclude that $cd' = dc'$. Multiplying these two equalities together, we have $ab'cd' = ba'dc'$ which we rewrite as $acb'd' = bda'c'$, so that $(ac, bd) \sim (a'c', b'd')$.

(c) Any two integers $z, z' \in \mathbb{Z}$ are identified with the equivalence classes $[(z, 1)], [(z', 1)]$ (respectively). Their product zz' is identified with $[(zz', 1)]$. For the multiplication operation to "work well", we should have $[(z, 1)] \otimes [(z', 1)] = [(zz', 1)]$. This is clear since

$$[(z, 1)] \otimes [(z', 1)] = [(z \cdot z', 1 \cdot 1)] = [(zz', 1)].$$

(d) We defined $\frac{a}{b}$ (with $b \neq 0$) to be $[(a, b)]$. Via our identification of z with $\frac{z}{1}$ or the equivalence class $[(z, 1)]$, the operation $z \times \frac{1}{z}$ is interpreted as

$$[(z, 1)] \otimes [(1, z)] = [(z \cdot 1, 1 \cdot z)] = [(z, z)].$$

Moreover, the integer 1 is identified with $[(1, 1)]$. Thus, to prove that $z \times \frac{1}{z} = 1$, we must show that $[(z, z)] = [(1, 1)]$, or $(z, z) \sim (1, 1)$. This is clear, since $z \cdot 1 = 1 \cdot z$.

(e) Almost any pair of elements of Q would work as a counter-example. For example,

$$[(1, 1)] \boxplus [(2, 1)] = [(1 + 2, 1 + 1)] = [(3, 2)].$$

On the other hand, $[(1,1)] = [(2,2)]$ because $(1,1) \sim (2,2)$ (that is, $1 \cdot 2 = 1 \cdot 2$). However,

$$[(2,2)] \boxplus [(2,1)] = [(2+2, 2+1)] = [(4,3)]$$

and $[(3,2)] \neq [(4,3)]$ (because $3 \cdot 3 \neq 2 \cdot 4$).

In summary, even though $[(1,1)] = [(2,2)]$ and $[(2,1)] = [(2,1)]$ we have

$$[(1,1)] \boxplus [(2,1)] \neq [(2,2)] \boxplus [(2,1)]$$

showing that $\boxplus$ is not well-defined.

(f) Suppose $[(a,b)] = [(a',b')]$ and $[(c,d)] = [(c',d')]$. To prove that $\oplus$ is well-defined we must prove that

$$[(a,b)] \oplus [(c,d)] = [(a',b')] \oplus [(c',d')].$$

By the definition of $\oplus$, the left side is $[(ad+bc, bd)]$ and the right side is $[(a'd'+b'c', b'd')]$, so we must show that $(ad+bc, bd) \sim (a'd'+b'c', b'd')$. That is, we need to prove that

$$(ad+bc)b'd' = bd(a'd'+b'c')$$

or that

$$ab'dd' + bb'cd' = a'bdd' + bb'c'd$$

Since $(a,b) \sim (a',b')$ we have $ab' = ba' = a'b$. Similarly, $cd' = dc' = c'd$. Using the first equation on the first summand and the second equation on the second we have

$$ab'dd' + bb'cd' = a'bdd' + bb'c'd$$

as we wanted to show.

(g) Any two integers $z, z' \in \mathbb{Z}$ are identified with the equivalence classes $[(z,1)], [(z',1)]$ (respectively). Their sum $z + z'$ is identified with $[(z+z',1)]$. For the addition operation to "work well", we should have $[(z,1)] \oplus [(z',1)] = [(z+z',1)]$. This is clear since

$$[(z,1)] \oplus [(z',1)] = [(z \cdot 1 + 1 \cdot z', 1 \cdot 1)] = [(z+z',1)].$$

(h) Suppose $[(a,b)] = [(a',b')]$. For $\div$ to be well-defined, we must show that $\div[(a,b)] = \div[(a',b')]$. The left side is $[(b,a)]$ and the right side $[(b',a')]$, so we must prove that $(b,a) \sim (b',a')$. Equivalently, we must show that $ba' = ab'$.

Since $[(a,b)] = [(a',b')]$, we have $ab' = ba'$, so we are done.

(i) Since (the unary operation) $\div$ is well-defined and $\otimes$ is well-defined, the binary operation $\div$ is also well-defined.

In more detail, suppose $[(a,b)] = [(a',b')]$ and $[(c,d)] = [(c',d')]$. We must show that

$$[(a,b)] \div [(c,d)] = [(a',b')] \div [(c',d')].$$

First we note that since $c \neq 0$ and $[(c,d)] = [(c',d')]$ we must have $c' \neq 0$ (otherwise $cd' \neq dc'$ since both $c, d' \neq 0$). Therefore, it makes sense to apply $\div$ to the right side.

Next, since $[(c,d)] = [(c',d')]$ and the unary operation $\div$ is well-defined, we know that $\div[(c,d)] = \div[(c',d')]$. Finally, since $\otimes$ is well-defined, we know that

$$[(a,b)] \otimes (\div[(c,d)]) = [(a',b')] \otimes (\div[(c',d')]) .$$

The right side is $[(a,b)] \div [(c,d)]$ and the left side $[(a',b')] \div [(c',d')]$.

(j) Let $a, b \in \mathbb{Z}$ with $b \neq 0$. These integers are identified with the equivalence classes $[(a,1)], [(b,1)]$ (respectively). Then,

$$\begin{aligned}[(a,1)] \div [(b,1)] &= [(a,1)] \otimes (\div[(b,1)]) \\ &= [(a,1)] \otimes [(1,b)] \\ &= [(a \cdot 1, 1 \cdot b)] = [(a,b)]\end{aligned}$$

which is precisely our interpretation of the symbol $\frac{a}{b}$.

More generally, the symbols $\frac{r}{s}, \frac{t}{u}$ are identified with the equivalence classes $[(r,s)], [(t,u)]$ (respectively). Then,

$$\begin{aligned}[(r,s)] \div [(t,u)] &= [(r,s)] \otimes (\div[(t,u)]) \\ &= [(r,s)] \otimes [(u,t)] \\ &= [(ru, st)]\end{aligned}$$

which is precisely our interpretation of the symbol $\frac{ru}{st}$.

(Exercise on page 132.)

Solution for Exercise 117 (Non Functions)**.**

(a) Note that d is in the domain of f, but there is no element y in the codomain (the set $\{1,2,3\}$), such that $(d,y) \in f$. Therefore, f is not a function.

(b) Note that $(b,2),(b,1) \in f$, but for f to be a function, there must be a unique $y \in \{1,2,3\}$ such that $(b,y) \in f$. Since there is no *unique* element in $\{1,2,3\}$ with that property (there are 2 distinct elements that satisfy this), f is not a function.

(c) $f : \mathbb{R} \to \mathbb{R}, f(x) = \frac{1}{x}$. Note that $0 \in \mathbb{R}$, but $f(0) = \frac{1}{0}$ is undefined. So there is no element $y \in \mathbb{R}$ such that $(0,y) \in f$. Therefore, f is not a function.

(d) $f : \mathbb{N} \to \mathbb{N}, f(x) = \sqrt{x}$. Note that for integers that are not perfect squares, for example 2, there is no element $y \in \mathbb{N}$ such that $(2,y) \in f$. Therefore, f is not a function.

(e) $f : \mathbb{R} \to \mathbb{R}, f(x) = \sqrt{x}$. Note that for negative numbers, for example -1, the expression $\sqrt{-1}$ is undefined in $\mathbb{R}$. In other words, there is no element $y \in \mathbb{R}$ such that $(-1,y) \in f$. Therefore, f is not a function.

(f) $f \subseteq \mathbb{R}$ is defined by $(x,y) \in f$ if and only if $x = |y|$ (the absolute value of y). Note that $(1,1),(1,-1) \in f$ (since $1 = |1|$ and also $1 = |-1|$), so there is no single $y \in \mathbb{R}$ such that $(1,y) \in f$ (there are 2 distinct such y).

(Exercise on page 133.)

Solution for Exercise 118 (Function Construction)**.**

(a) There are 4 possible relations from A to B:

$$R_0 = \emptyset$$
$$R_1 = \{(1, a)\}$$
$$R_2 = \{(1, b)\}$$
$$R_3 = \{(1, a); (1, b)\}$$

Of these, only R_1 and R_2 are functions.

(b) There are 9 possible functions from C to D:

$$f_1 = \{(1, a); (2, a)\} \qquad f_4 = \{(1, b); (2, a)\} \qquad f_7 = \{(1, c); (2, a)\}$$
$$f_2 = \{(1, a); (2, b)\} \qquad f_5 = \{(1, b); (2, b)\} \qquad f_8 = \{(1, c); (2, b)\}$$
$$f_3 = \{(1, a); (2, c)\} \qquad f_6 = \{(1, b); (2, c)\} \qquad f_9 = \{(1, c); (2, c)\}$$

(c) To construct a function $f : A \to B$, we must pick exactly one output in B for each input in A. So we will have exactly 3 pairs in our relation. Two examples are

$$f_1 = \{(1, a); (2, b); (3, c)\} \qquad \text{and} \qquad f_2 = \{(1, a); (2, a); (3, a)\}.$$

In order for a relation $R \subseteq A \times B$ to be a function, it must satisfy two conditions:

- For every $x \in A$, there exists $y \in B$ such that $(x, y) \in R$.
- If $(x, y_1), (x, y_2) \in R$, then $y_1 = y_2$ (this is the uniqueness condition).

We construct a relation R_1 which violates the first condition but satisfies the second, and a relation R_2 which violates the second condition but satisfies the first.

$$R_1 = \{(1, a); (2, a)\} \qquad \text{and} \qquad R_2 = \{(1, a); (1, b); (2, a); (3, c)\}.$$

(d) To construct a function from A to B, for each element of A, we must choose exactly one element from B. So for the element $1 \in A$, we have 3 possible choices from $B = \{a, b, c\}$ for the image of x. For *each* one of these choices we have 3 distinct choices for the image $f(2)$, so that in total we have $3 \cdot 3 = 9$ choices for $f(1)$ and $f(2)$. For *each* one of these nine choices, we have 3 distinct choices for the image $f(3)$, so that in total we have $3 \cdot 9 = 27$ distinct functions from A to B.

(e) The same reasoning as in the previous part shows that we have n^m functions $M \to N$ (we have n options for each $m \in M$) and m^n functions $N \to M$.

(Exercise on page 134.)

Solution for Exercise 119 (Is This a Function?)**.**

(a) The digraph represents a function since each element in the domain $\{a, b, c\}$ is related to exactly one element in the codomain $\{1, 2, 3, 4\}$.

(b) The relation is not a function because the element 2 in the domain is not related to any element in the codomain $\mathbb{R}$.

(c) The relation f is a function, since each integer x in the domain is related to exactly one integer $x^2 + 1$ in the codomain.

(d) The graph does not represent a function. Note that the graph does not pass the vertical line test, signifying that there are elements in the domain that are related to more than one distinct element in the codomain. For example, $(0, 1)$ and $(0, -1)$ are both points on the graph (the northernmost and southernmost points on the circle).

(e) The relation S is a function because each natural number n in the domain is related to exactly one natural number $n + 1$ in the codomain.

(f) The relation represented by the digraph is not a function since the element c is related to two distinct elements 2 and 3.

(g) The relation g is a function, since each integer n in the domain is related to exactly one integer in the codomain, namely the number of digits that make up the integer n.

(h) The relation is a function because each element in the domain $\{1, 2, 3, 4\}$ is related to exactly one element in the codomain.

(Exercise on page 135.)

Solution for Exercise 120 (Domain and Range)**.**

(a) The domain is the set of nonnegative integers, so $\mathrm{Dom}(f) = \mathbb{Z}_{\geq 0}$. The outputs are the 10 digits from 0 to 9, so $\mathrm{Rng}(f) = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

(b) The domain is the set of English letters, so $\mathrm{Dom}(f) = \{a, b, c, \ldots, x, y, z\}$. The outputs consist of the position of each letter, with $f(a) = 1$ and $f(z) = 26$, so $\mathrm{Rng}(f) = \{1, 2, \ldots, 26\}$.

(c) The domain is the set of pairs of positive integers, so $\mathrm{Dom}(f) = \mathbb{N} \times \mathbb{N}$. Since each positive integer can be an output (specifically $\max\{n, n\} = n$), the range is $\mathrm{Rng}(f) = \mathbb{N}$.

(d) The domain is the set of finite sequences of 0 and 1, $\mathrm{Dom}(f) = \{b \mid b \text{ is a sequence of 0's and 1's}\}$. Since such a sequence can have any number of 0's (including no 0's), the range is $\mathrm{Rng}(f) = \mathbb{Z}_{\geq 0}$.

(e) The domain is the set of real numbers, $\mathrm{Dom}(f) = \mathbb{R}$. A square of a real number is necessarily nonnegative, and each nonnegative real r is an output (since $(\pm\sqrt{r})^2 = r$), so $\mathrm{Rng}(f) = \mathbb{R}_{\geq 0}$.

(Exercise on page 136.)

Solution for Exercise 121 (Codomain versus Range).

(a) $\mathrm{Dom}(f) = \mathbb{R}$, and $\mathrm{Codom}(f) = \mathbb{R}$; these are part of the definition of f. The range of f is the set of actual outputs of f; since $x \in \mathbb{R} \implies x^2 \geq 0$, we have $x^2 + 1 \geq 1$. Conversely, if $y \geq 1$ then $x = \sqrt{y-1}$ is a well-defined real number and $x^2 = y$. In summary, $\mathrm{Rng}(f) = [1, \infty)$. Note that the codomain and range of f are not equal. For example, we have $-1 \in \mathrm{Codom}(f)$, but $-1 \notin \mathrm{Rng}(f)$.

(b) $\mathrm{Dom}(g) = \mathbb{R}$, and $\mathrm{Codom}(g) = [1, \infty)$; these are part of the definition of g. Exactly the same analysis as for f shows that the range of g is $\mathrm{Rng}(g) = [1, \infty)$. Note that in this case the codomain of g is equal to the range of g.

(c) As we saw with f and g, two functions can have the same rule, but because their specified codomains are different, they are *different functions*. That is, $f \neq g$. This is a very important distinction, which will play a significant role in §8.2.

Another way that two functions with the same rule may be different is if their domain is different. For example, $h : \mathbb{N} \to \mathbb{N}$ given by $x \mapsto x^2$ and $j : \mathbb{Z} \to \mathbb{N}$ given by $x \mapsto x^2$ are different functions. The important things to remember is that the domain and codomain are part of the *definition* of a function and must be specified (sometimes implicitly or by convention).

(Exercise on page 137.)

Solution for Exercise 122 (Special Functions).

(a) Let's write ι as a relation, $\iota = \{(1,1);(2,2);(3,3);(4,4)\}$. We can see that each element in A is related to exactly one element in B, therefore, ι is a function.

We cannot define $\iota : B \to A$ using the same rule, because there are elements in B that are not in A. For example, what would $\iota(8)$ be equal to? According to the rule, $\iota(8) = 8$, but $8 \notin A$, and A is supposed to be the codomain in this case.

(b) The identity map and the inclusion map have the same rule. But they have different codomains making them different functions. The codomain of ι is B, while the codomain of i_A is A.

(c) The domain of c is specified as A, and the codomain is specified as B. The range of c is the actual outputs of c, and for every input in A, the output of the function is 6, therefore, $\text{Rng}(c) = \{6\}$.

(d) We would run into the same problem we did when trying to reverse the direction of ι. Consider $1 \in A$, we would have $c(1) = -1$, but $-1 \notin \mathbb{N}$ and $\mathbb{N}$ is the specified codomain of c. Therefore, the function would not be well-defined.

(Exercise on page 138.)

Solution for Exercise 123 (Piecewise-Defined Functions).

(a) Note that $x = 1$ satisfies the first two conditions:

- Since $0 < x = 1 \leq 1$, we have $f(1) = 1^2 = 1$.
- Since $x = 1 \geq 1$, we have $f(1) = 2(1) + 1 = 3$

Since $1 \neq 3$, the function f is not well-defined; there are 2 distinct outputs assigned to $f(1)$.

In contrast, with $f(0)$ we run into a different problem. Note that $x = 0$ is not in any of the ranges specified in the rule for f,

- $x = 0$ does not satisfy $0 < x \leq 1$;
- $x = 0$ does not satisfy $x \geq 1$;
- $x = 0$ does not satisfy $x < 0$.

Since $0 \in \mathbb{R}$ is an element of the domain, it must be assigned a value. This is another reason (distinct from the first) that f is not a well-defined function.

(b) There are infinitely many ways to revise the conditions in the definition of f to make the function well-defined. One of the simplest ways of doing so is

$$g(x) = \begin{cases} x^2 & \text{if } 0 < x \leq 1 \\ 2x + 1 & \text{if } x > 1 \\ 3x - 4 & \text{if } x \leq 0. \end{cases}$$

We claim that g is now a function. Indeed, the conditions cover $\mathbb{R}$, since

$$(-\infty, 0] \cup (0, 1] \cup (1, \infty) = \mathbb{R}$$

ensuring that every element of the domain is assigned an element in the range. (We are implicitly assuming that it is obvious that x^2, $2x + 1$, and $3x - 4$ result in real numbers; but this of course needs to be verified). Moreover, the pieces are pairwise disjoint:

$$(-\infty, 0] \cap (0, 1] = \emptyset; \qquad (-\infty, 0] \cap (1, \infty) = \emptyset; \qquad (0, 1] \cap (1, \infty) = \emptyset.$$

Therefore, no element of the domain is assigned more than one value.

(c) Suppose $h : A \to B$ is a piecewise-defined function

$$h(x) = \begin{cases} b_1 & \text{if } x \in A_1; \\ b_2 & \text{if } x \in A_2; \\ b_3 & \text{if } x \in A_3. \end{cases}$$

We must ensure that every element in the domain is mapped to exactly one element in the codomain. Therefore, we must check that

- $A_1 \cup A_2 \cup A_3 = A$, this ensures that every element in the domain is mapped to *at least one* element in the codomain.
- Next, we must ensure that every element in the domain is mapped to *at most one* element in the codomain. We must ensure that if $A_i \cap A_j \neq \emptyset$ then $b_i = b_j$.

  One way this can happen is if the hypothesis is always false, i.e. if $A_i \cap A_j = \emptyset$ for every $i \neq j$ (so A_1, A_2, A_3 are pairwise disjoint). However, another way this can happen is if the consequent is always true, i.e. $b_1 = b_2 = b_3$. (There are other possibilities too.)

(Exercise on page 139.)

Solution for Exercise 124 (The Ceiling and Floor Functions)**.**

(a) Let's take a look at each one in turn.

$$\begin{aligned} \lceil 2.1 \rceil &= \text{ the least integer greater than or equal to } 2.1 = 3, \\ \lfloor 2.1 \rfloor &= \text{ the greatest integer less than or equal to } 2.1 = 2, \\ \lceil -2.1 \rceil &= \text{ the least integer greater than or equal to } -2.1 = -2, \\ \lfloor -2.1 \rfloor &= \text{ the greatest integer less than or equal to } -2.1 = -3. \end{aligned}$$

(b) The ceiling function rounds a number up to the nearest integer, while the floor function rounds a number down to the nearest integer.

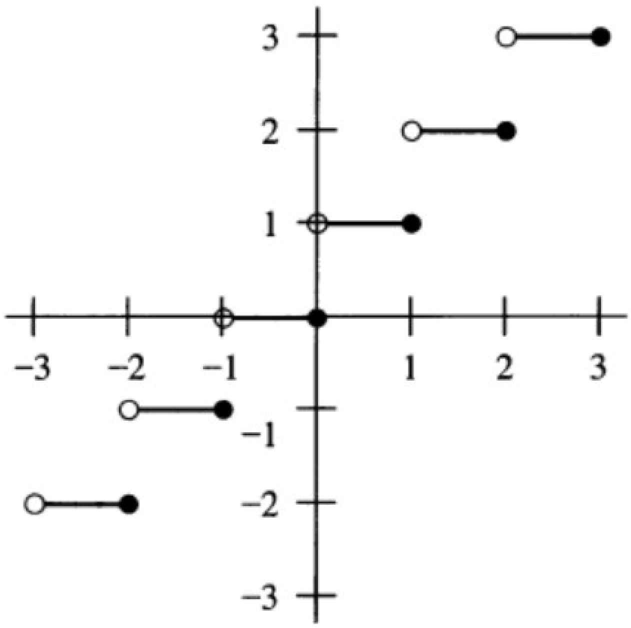


Figure 21.7: Graph for $\lceil x \rceil$

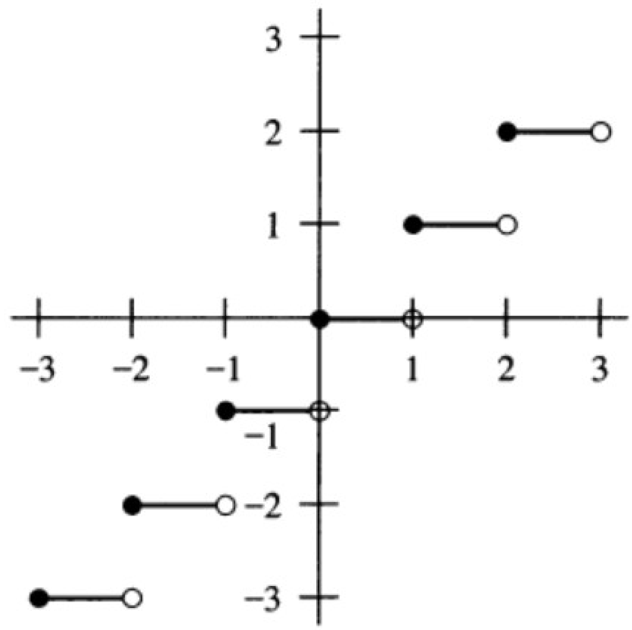


Figure 21.8: Graph for $\lfloor x \rfloor$

(c) Note that for the ceiling functions, the leftmost point of each line segment is not included in the graph, while the rightmost point is. The graph jumps up at each integer (so that the output for inputs between two consecutive integers is always the larger integer).

Dually, for the floor function, the leftmost point of each line segment is included in the graph, while the rightmost point is not. For the floor function, the output for an input between two consecutive integers is always the smaller integer

(d) If we define the ceiling function by

$$\lceil x \rceil = \text{ an integer greater than or equal to } x,$$

It would not be well-defined since there are infinitely many integers greater than or equal to any real number. For example, for this definition, we would have $\lceil 1 \rceil = 1, 2, 3, \ldots$. Therefore, this rule would not satisfy the definition of a function.

(e) Let $\theta = r - \lfloor r \rfloor$. By the definition of the floor function, $\lfloor r \rfloor \leq r$, so that $\theta \geq 0$.

Suppose towards contradiction that $\theta \geq 1$, then from $1 \leq \theta = r - \lfloor r \rfloor$ we have $\lfloor r \rfloor + 1 \leq r$. But $\lfloor r \rfloor + 1$ is an integer strictly greater than $\lfloor r \rfloor$, contradicting the fact that $\lfloor r \rfloor$ is the greatest integer less than or equal to r. This contradiction proves that $\theta < 1$.

In conclusion, $\theta \in [0, 1)$.

(f) Let $n = \lfloor r \rfloor$ and $\theta = r - \lfloor r \rfloor$. It is clear that $r = n + \theta$ and it remains to prove uniqueness.

Suppose that $r = n + \theta = n' + \theta'$ where $n, n' \in \mathbb{Z}$ and $\theta, \theta' \in [0, 1)$. Rearranging we have $n - n' = \theta' - \theta$. Since $\theta, \theta' \in [0, 1)$ we have $\theta - \theta' \in (-1, 1)$. On the other hand, $n - n' \in \mathbb{Z}$. Because the only integer in $(-1, 1)$ is 0, we conclude that $n - n' = \theta - \theta' = 0$ so that $n = n'$ and $\theta = \theta'$. This proves that the representation of r is unique.

(Exercise on page 140.)

Solution for Exercise 125 (Functions and Equivalence Relations)**.**

(a) The function is not well-defined. For example, $[0] = [10]$, but $f([0]) = 0 \neq 10 = f([10])$.

(b) The function is well-defined, it is just the identity function on $\mathbb{Z}/\equiv_{10}$. Indeed, if $[x] = [y]$ then $f([x]) = [x] = [y] = f([y])$.

(c) The function is well-defined. Suppose $[x] = [y]$, so that $10|y-x$. It follows that $y-x$ ends in 0 and therefore that x, y have the same last digit (cf. Exercise 6 in the Equivalence Relation handout). Therefore, $f([x]) = f([y])$.

(d) The function is well-defined. If $[(x, y)] = [(x', y')]$, we have $(x, y) \sim (x', y')$ so that $x = x'$. Therefore, $f[(x, y)] = x = x' = f([(x', y')])$.

(e) The function is not well-defined. For example, $[(0, 0)] = [(0, 1)]$, since $(0, 0) \sim (0, 1)$. However, $f([(0, 0)]) = 0 \neq 1 = f[(0, 1)]$.

(f) The function is well-defined. Suppose $[x] = [y]$ so that $y - x \in \mathbb{Z}$. Let us write $x = \lfloor x \rfloor + \theta_x$ and $y = \lfloor y \rfloor + \theta_y$ with $\theta_x, \theta_y \in [0, 1)$ (see Exercise 124(f)); we claim that $\theta_x = \theta_y$. Indeed,

$$y - x = (\lfloor y \rfloor - \lfloor x \rfloor) + (\theta_y - \theta_x).$$

Since $y - x$ and $\lfloor y \rfloor - \lfloor x \rfloor$ are integers, so is $\theta_y - \theta_x$

$$\theta_y - \theta_x = (y - x) - (\lfloor y \rfloor - \lfloor x \rfloor).$$

Since $\theta_x, \theta_y \in [0, 1)$, we have $\theta_y - \theta_x \in (-1, 1)$ and the only integer in this interval is 0, proving that $\theta_x = \theta_y$.

It now follows that $f([x]) = \theta_x = \theta_y = f([y])$.

(Exercise on page 141.)

Solution for Exercise 126 (Some Properties of the Ceiling and Floor Functions)**.**

(a) Following Exercise 124(f), we use the unique expression $x = n + \theta$ with $n \in \mathbb{Z}$ and $\theta \in [0, 1)$. We distinguish between the cases that θ is greater or less than $\frac{1}{2}$.

- Suppose $0 \leq \theta < \frac{1}{2}$. Then,
$$2x = 2n + 2\theta$$
is the unique expression of $2x$, since $2\theta \in [0, 1)$. Therefore, $2n$ is the greatest integer less than or equal to $2x$, so that $\lfloor 2x \rfloor = 2n$.
Similarly, since $\theta + \frac{1}{2} \in [0, 1)$, we have
$$x + \frac{1}{2} = \lfloor x \rfloor + (\theta + \frac{1}{2})$$
as the unique expression of $x + \frac{1}{2}$ so that $\lfloor x + \frac{1}{2} \rfloor = \lfloor x \rfloor = n$.
In conclusion,
$$\lfloor x \rfloor + \left\lfloor x + \frac{1}{2} \right\rfloor = n + n = 2n = \lfloor 2x \rfloor.$$

- Suppose $\frac{1}{2} \leq \theta < 1$. In this case $2\theta \in [1, 2)$ so that $(2\theta - 1) \in [0, 1)$ and
$$2x = 2n + 1 + (2\theta - 1)$$
is the unique expression of $2x$. It follows that $2n + 1$ is the greatest integer less than or equal to $2x$, so that $\lfloor 2x \rfloor = 2n + 1$.
Similarly,
$$x + \frac{1}{2} = n + 1 + \left(\theta + \frac{1}{2} - 1\right).$$
so that $\left\lfloor x + \frac{1}{2} \right\rfloor = n + 1$.
In conclusion,
$$\lfloor x \rfloor + \left\lfloor x + \frac{1}{2} \right\rfloor = n + (n + 1) = 2n + 1 = \lfloor 2x \rfloor.$$

(b) The statement is false. There are infinitely many counterexamples, one of which is $x = y = \frac{1}{2}$. Then,
$$\lceil x + y \rceil = \left\lceil \frac{1}{2} + \frac{1}{2} \right\rceil = \lceil 1 \rceil = 1 \neq 2 = 1 + 1 = \left\lceil \frac{1}{2} \right\rceil + \left\lceil \frac{1}{2} \right\rceil = \lceil x \rceil + \lceil y \rceil.$$

(c) The statement is false. The same counterexample as above would work here as well: for $x = y = \frac{1}{2}$, we have
$$\lfloor x + y \rfloor = \left\lfloor \frac{1}{2} + \frac{1}{2} \right\rfloor = \lfloor 1 \rfloor = 1 \neq 0 = 0 + 0 = \left\lfloor \frac{1}{2} \right\rfloor + \left\lfloor \frac{1}{2} \right\rfloor = \lfloor x \rfloor + \lfloor y \rfloor.$$

(d) Write $x = n + \theta$ with $n \in \mathbb{Z}$ and $\theta \in [0, 1)$. We distinguish between the cases that n is even or odd.

- If n is odd, say $n = 2k + 1$, then $\frac{x}{2} = k + (\frac{1}{2} + \frac{\theta}{2})$ is the unique expression of $\frac{x}{2}$. (Note that since $\theta \in [0, 1)$, we have $\frac{1}{2} + \frac{\theta}{2} \in [0, 1)$.) We therefore have $\lfloor x/2 \rfloor = k$ and $\lceil x/2 \rceil = k + 1$, so that
$$\left\lfloor \frac{x}{2} \right\rfloor + \left\lceil \frac{x}{2} \right\rceil = k + (k + 1) = 2k + 1 = \lfloor x \rfloor.$$

- If n is even, say $n = 2k$, then $\frac{x}{2} = k + \frac{\theta}{2}$ is the unique expression of $\frac{x}{2}$. Therefore, $\left\lfloor \frac{x}{2} \right\rfloor = k$. The value of $\left\lceil \frac{x}{2} \right\rceil$ depends on the value of θ.
  - If $\theta = 0$, then $\left\lceil \frac{x}{2} \right\rceil = k$. Similarly, $\lceil x \rceil = \lceil 2k \rceil = 2k$. Therefore,
  $$\left\lfloor \frac{x}{2} \right\rfloor + \left\lceil \frac{x}{2} \right\rceil = 2k = \lceil x \rceil.$$
  - If $\theta > 0$, then $\lceil x/2 \rceil = k + 1$. Similarly, $\lceil x \rceil = \lceil 2k + \theta \rceil = 2k + 1$. Therefore,
  $$\left\lfloor \frac{x}{2} \right\rfloor + \left\lceil \frac{x}{2} \right\rceil = k + (k+1) = 2k + 1 = \lceil x \rceil.$$

  We see that in either case, we have $\lfloor x/2 \rfloor + \lceil x/2 \rceil = \lceil x \rceil$.

In conclusion,
$$\left\lceil \frac{x}{2} \right\rceil + \left\lfloor \frac{x}{2} \right\rfloor = \begin{cases} \lfloor x \rfloor & \text{if } \lfloor x \rfloor \text{ is odd;} \\ \lceil x \rceil & \text{if } \lfloor x \rfloor \text{ is even.} \end{cases}$$

(Exercise on page 142.)

Solution for Exercise 127 (Basic definitions).

(a) None of the above; in fact, any function satisfies this condition. For example, the constant function $c : \mathbb{R} \to \mathbb{R}$ mapping $x \mapsto 1$ satisfies the condition but is neither injective nor surjective.

(b) This is the definition of surjectivity.

(c) None of the above. The same example in part (a) works here as well.

(d) None of the above; again, any functions satisfies this condition (in particular, the same example as in part (a) works here as well).

(e) This condition guarantees bijectivity. Indeed, surjectivity follows from "at least one x" and injectivity follows from "at most one x".

(f) This is the definition of injectivity.

(g) This condition is equivalent to surjectivity.

(h) This condition, being the contrapositive of part (f) above, is one of the equivalent definitions of injectivity.

(i) This condition is also one of the equivalent definitions of injectivity. This is because the "if" part is obvious from the definition of a function (it's part of what it means for a function to be well-defined).

(Exercise on page 143.)

Solution for Exercise 128 (Classifying functions)**.**

(a) The function f is bijective.

To prove injectivity, suppose $f(x) = f(x')$ so that $3x + 2 = 3x' + 2$ and it follows that $x = x'$.

To prove surjectivity, note that $f\left(\frac{y-2}{3}\right) = y$.

(b) The function g is neither injective nor surjective.

Indeed, $f(-1) = f(1)$ even though $1 \neq -1$, proving that the function is not injective.

Moreover, there is no real number $r \in \mathbb{R}$ such that $f(r) = r^2 = -1$, proving that the function is not surjective.

(c) The function h is bijective.

To prove injectivity, suppose $h(x) = h(x')$ so that $x^2 = (x')^2$. It follows that $x = \pm x'$, and since $x, x' \in [0, \infty)$, we must have $x = x'$.

To prove surjectivity, note that for any $y \in [0, \infty)$ we have $f(\sqrt{y}) = y$.

(d) The function k is surjective but not injective.

Indeed, $k(-1) = k(1) = 1$, proving the function is not injective. On the other hand, for any $y \in [0, \infty)$ we have $k(\sqrt{y}) = y$, proving that k is surjective.

Helpful Tip!

Pay special attention to the difference between the functions g, h, and k. Injectivity and surjectivity crucially depend on the domain and codomain of the function, as well as on the rule or matching process.

(e) The function p is injective but not surjective.

To prove injectivity, suppose $p(n) = p(n')$ so that $n + 1 = n' + 1$, it follows that $n = n'$.

On the other hand, there is no $n \in \mathbb{N}$ such that $p(n) = 1$, because for every $n \in \mathbb{N}$ we have $n \geq 1$ so that $p(n) = n + 1 \geq 2$. This shows that p is not surjective.

(Exercise on page 144.)

Solution for Exercise 129 (Piecewise-defined Function).

(a) Note that h is well-defined because $A \cap C = \emptyset$. Indeed, we can prove that for every $x \in A \cup C$ there is exactly one $y \in B \cup D$ with $h(x) = y$ as follows:

- If $x \in A$, then there is exactly one $y \in B$ such that $f(x) = y$ and $h(x) = f(x) = y$;
- If $x \in C$, then there is exactly one $y \in D$ such that $g(x) = y$ and $h(x) = g(x) = y$.

Since no x is in both A and C, there is exactly one y such that $h(x) = y$ (and that y is either $f(x)$ or $g(x)$). (See also Exercise 7 in the Introduction to Functions handout.)

(b) Even if f, g are injective, it does not necessarily follow that h is injective. For example, consider the functions

$$f : \{a, b\} \to \{1, 2\}, \qquad g : \{c, d\} \to \{1, 2\}$$
$$a \mapsto 1, b \mapsto 2. \qquad c \mapsto 1,\ d \mapsto 2.$$

Then, (the domains of f and g are disjoint and) f, g are injective, but the piecewise function h is not! Indeed, $h(a) = 1 = h(c)$.

(c) If f, g are surjective then so is h. To see this, let $y \in B \cup D$ be arbitrary; then $y \in B$ or $y \in D$ (or both).

- Suppose $y \in B$. Since $f : A \to B$ is surjective, there is some $a \in A$ such that $f(a) = y$. Then $h(a) = f(a) = y$.
- Suppose $y \in D$. Since $g : C \to D$ is surjective, there is some $c \in C$ such that $g(c) = y$. Then $h(c) = g(c) = y$.

Either way, we have found some $x \in A \cup C$ such that $h(x) = y$, proving that h is surjective.

(d) Suppose $B \cap D = \emptyset$ and f, g are bijective, then h is also bijective. Since f, g are surjective, so is h (from the previous part), so it remains to prove that h is injective. We prove more generally that if $B \cap D = \emptyset$ and f, g are injective, then so is h.

Suppose $x, x' \in A \cup C$ are such that $h(x) = h(x')$. Since the domain of h is $A \cup C$, we know that each of x, x' is an element of A or of C. We claim that both x, x' must belong to the same set.

Indeed, suppose towards contradiction that one of x, x' is an element of A and the other of C. Without loss of generality, suppose $x \in A$ and $x' \in C$. Now, $h(x) = f(x) \in B$ (since the range of f is B), and $h(x') = g(x) \in D$ (since the range of g is D). Then $h(x) = h'(x) \in B \cap D$, contradicting the assumption that $B \cap D = \emptyset$. This contradiction proves that x, x' must belong to the same set (either A or C).

- If $x, x' \in A$, then $f(x) = h(x) = h(x') = f(x')$ and from the injectivity of f we conclude $x = x'$.
- If $x, x' \in C$, then $g(x) = h(x) = h(x') = g(x')$ and from the injectivity of g we conclude $x = x'$.

Either way, we see that $x = x'$, concluding the proof that h is injective.

(Exercise on page 145.)

Solution for Exercise 130 (Finite sets).

Let $m, n \in \mathbb{N}$ and set $A = \{1, 2, \ldots, n\}$ and $B = \{1, 2, \ldots, m\}$.

(a) Note that if $n \leq m$, then $A \subseteq B$, so the inclusion map $\iota : A \to B$ given by $x \mapsto x$ is an injection.[29]

(b) We define $f : A \to B$ by

$$f(x) = \begin{cases} x & \text{if } x \leq m; \\ 1 & \text{if } x > m. \end{cases}$$

Since the conditions are mutually exclusive, this is a well-defined function, and it is clearly surjective in the case that $n \geq m$.

(c) Note that if $n = m$ then $A = B$, so the identity function $i : A \to B$ is a bijection.[30]

(d) Since f is a surjection, for each $b \in B$ there is at least one $a \in A$ such that $f(a) = b$. In other words,

$$B \subseteq \bigcup_{a \in A} \{f(a)\}.$$

That is, each $b \in B$ appears in at least one of the sets $\{f(a)\}$. It follows that the total number of elements of B is *at most* the total number of elements in all of the sets $\{f(a)\}$. Since each $\{f(a)\}$ has exactly one element, this number is at most the size of the index set A. In symbols,

$$m = |B| = \left| \bigcup_{a \in A} \{f(a)\} \right| \leq \sum_{a \in A} |\{f(a)\}| = \sum_{a \in A} 1 = n.$$

(e) There are different ways to approach this proof. A rather fast proof proceeds via the pigeonhole principle, but we shall present a slightly different (but equivalent) approach which anticipates the definition of preimages, which we shall discuss in an upcoming handout.

Suppose $g : A \to B$ is an injection. For each $b \in B$, there can be at most one element $a \in A$ with $g(a) = b$ (cf. Exercise 127 and the corresponding hint). Let us define S_b as the set $\{a\}$ if there is such an element with $g(a) = b$, otherwise $S_b = \emptyset$.

Since g is a function, every $x \in A$ is mapped to at least one $y \in B$, so that

$$A = \bigcup_{b \in B} S_b$$

(make sure you understand what this equality claims and why it is true). In other words, each $x \in A$ is in at least one S_b. Therefore, the total number of elements in A is at most the total number of elements in all of the S_b, i.e.

$$n \leq \sum_{b \in B} |S_b|.$$

But each $|S_b| \in \{0, 1\}$, so that

$$\sum_{b \in B} |S_b| \leq \sum_{b \in B} 1 = m$$

proving that $n \leq m$.

[29] You were asked to prove this as Theorem 8.38 from the recommended reading, but it's a good exercise to reprove this if you do not remember the proof!

[30] Again, you were asked to prove this as Theorem 8.39, but it also follows immediately from the previous two parts (do you see why?).

(f) Since $h : A \to B$ is injective, we know from the previous part that $n \leq m$. Since $h : A \to B$ is surjective, we know from part (d) above that $m \leq n$. Combining these two inequalities, we conclude that $m = n$.

(g) All of these assertions follow from the various parts of the question.

- If follows from parts (a) and (e) that $n \leq m$ if and only if there exists an injection $A \to B$;
- It follows from parts (b) and (d) that $n \geq m$ if and only if there exists a surjection $A \to B$;
- It follows from parts (c) and (f) that $n = m$ if and only if there exists a bijection $A \to B$.

(Exercise on page 146.)

Solution for Exercise 131.

(a) The first element of row r is $(r-1)n+1$.

(b) The element (r,k) is the k-th element of row r. Since the first element $(r,1)$ has number $(r-1)n+1$, the k-th element is simply $(r-1)n+k$.

(c) Define $\Phi : A \times B \to \{1,\ldots,mn\}$ via $\Phi((r,k)) = (r-1)n+k$.

(d) To show that Φ is injective, suppose $\Phi((r,k)) = \Phi((r',k'))$, so that $(r-1)n+k = (r'-1)n+k'$. Rearranging and taking absolute values,

$$|r-r'|n = |k'-k|.$$

Now, $k,k' \in B$ so that $1 \le k,k' \le n$ and therefore $|k'-k| < n$. Since $|r-r'| \ge 0$, it follows that the only way the equality above can hold is if $|r-r'| = 0$ so that $|k'-k| = 0$. But this means that $r = r'$ and $k = k'$, proving that Φ is injective.

(e) To show that Φ is surjective, let $1 \le y \le mn$. Using Division with Remainder (see Exercise 6 in the Well Ordering Principle Handout), we can write $y = rn+q$ with $0 \le q < n$. Since $1 \le y \le mn$ we also must have $0 \le r \le m$.

- If $q = 0$ then $r > 0$, so that $(r,n) \in A \times B$ and we have $\Phi((r,n)) = (r-1)n+n = rn = y$.
- If $q > 0$ then $r < m$, so that $(r+1,q) \in A \times B$ and we have $\Phi((r+1,q)) = rn+q = y$.

This proves that every $y \in \{1,2,3,\ldots,mn\}$ has some $x \in A \times B$ such that $\Phi(x) = y$, so that Φ is surjective.

(Exercise on page 147.)

Solution for Exercise 132 (Set difference).

(a) Let $A, B \in X$ be arbitrary. To prove that $f[A] \setminus f[B] \subseteq f[A \setminus B]$, start with some $y \in f[A] \setminus f[B]$.

By the definition of set difference, $y \in f[A]$ and $y \notin f[B]$. By the definition of image, there exists some $a \in A$ such that $f(a) = y$ but for every $b \in B$ we have $f(b) \neq y$; in particular, $a \notin B$ and therefore $a \in A \setminus B$. Therefore, $y = f(a) \in f[A \setminus B]$.

(b) For an example where the inclusion is strict, consider the constant function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 1$. Almost any choice of A, B would work, for instance $A = \{1, 2\}$ and $B = \{3\}$. Then,

$$f[A] \setminus f[B] = \{1\} \setminus \{1\} = \emptyset.$$

On the other hand,

$$f[A \setminus B] = f[A] = \{1\}$$

and we have the strict inclusion $\emptyset \subsetneq \{1\}$.

(c) For the "if" direction, suppose for any $A, B \subseteq X$ we have

$$f[A \setminus B] = f[A] \setminus f[B].$$

To prove that f is injective, suppose $x, x' \in X$ are such that $x \neq x'$. Let $A = \{x\}$ and $B = \{x'\}$. Then $A \setminus B = A$ and therefore $f[A \setminus B] = f[A] = \{f(x)\}$. Therefore,

$$\{f(x)\} = f[A \setminus B] = f[A] \setminus f[B] = \{f(x)\} \setminus \{f(x')\}$$

so that we must have $f(x') \notin \{f(x)\}$, i.e. $f(x) \neq f(x')$. This proves that f is injective.

Conversely, suppose f is injective. We wish to show that

$$f[A \setminus B] \subseteq f[A] \setminus f[B].$$

Towards that end, let $y \in f[A \setminus B]$ be arbitrary (if the image is empty then the inclusion is trivial). By the definition of the image, there exists some $a \in A \setminus B$ such that $f(a) = y$. We claim there is no $b \in B$ such that $f(b) = y$.

Indeed, assume for contradiction we had some $b \in B$ with $f(b) = y$. Since $a \in A \setminus B$ (meaning $a \notin B$) we must have $a \neq b$. But then $f(a) = y = f(b)$, contradicting the hypothesis that f is injective. Therefore, for any $b \in B$ we have $f(b) \neq y$. That is, $y \notin f[B]$. Since $y = f(a) \in f[A]$, we conclude that $y \in f[A] \setminus f[B]$.

(Exercise on page 148.)

Solution for Exercise 133 (Cantor's Theorem)**.**

Assume for contradiction that for some $x \in X$ we have $f(x) = Y$. We consider the two possibilities of whether $x \in Y$ or $x \notin Y$.

- Suppose $x \in Y$. Then $x \in f(x)$ (since $Y = f(x)$). Then x does not satisfy the definition of Y (since $Y = \{x \in X : x \notin f(x)\}$) and so we must have $x \notin Y$, which is a contradiction.
- Suppose $x \notin Y$. Then $x \notin f(x)$ (since $Y = f(x)$). Then x satisfies the definition of Y (since $Y = \{x \in X : x \notin f(x)\}$) and so we must have $x \in Y$, which is a contradiction.

Either case $x \in Y$ and $x \notin Y$ leads to a contradiction; since these are all the possible cases, we obtain a contradiction to the assumption that for some $x \in X$ we have $f(x) = Y$. This contradiction proves there is no surjective function $X \to \mathcal{P}(X)$.

(Exercise on page 149.)

Solution for Exercise 134 (Composition of functions)**.**

(a) Note that the codomain of f is a subset of the domain of g, so that $g \circ f$ exists as a function $g \circ f : \mathbb{Z} \to \mathbb{Z}$. Similarly, the codomain of g is a subset of the domain of f, so that $f \circ g$ exists as a function $f \circ g : \mathbb{Z} \to \mathbb{Z}$. It remains to determine the rule in each case.

We determine the rule in each case. Given an arbitrary $x \in \mathbb{Z}$,

$$(f \circ g)(x) = f(g(x)) = f(3x^3) = 3x^3 + 2.$$

and

$$(g \circ f)(x) = g(f(x)) = g(x+2) = 3(x+2)^3 = 3(x^3 + 6x^2 + 12x + 8) = 3x^3 + 18x^2 + 36x + 24.$$

(b) For exactly the same reasons as in the previous problem, $f \circ g : \mathbb{Z}_6 \to \mathbb{Z}_6$ and $g \circ f : \mathbb{Z}_6 \to \mathbb{Z}_6$ are well-defined functions. It remains to find the rule.

Let $[x]_6 \in \mathbb{Z}_6$ be an arbitrary element. We use the fact that algebraic operations on representatives of equivalence classes modulo 6 are-well defined (see §7.4 of the recommended reading). Then,

$$(f \circ g)([x]_6) = f(g([x]_6)) = f([3x]_6) = [9x^2]_6 = [3x^2]_6.$$

(Note that $9x^2$ and $3x^2$ represent the same equivalence class modulo 6). Similarly,

$$(g \circ f)([x]_6) = g(f([x]_6)) = g([x^2]_6) = [3(x^2)]_6 = [3x^2]_6.$$

For these two functions we find $f \circ g = g \circ f$ (i.e. they *commute*); though this equality does not hold in general, as part (a) demonstrates.

(c) The codomain of g is not a subset of the domain of f, so the composition $f \circ g$ is undefined. For instance $g(2) = \sqrt{2}$ is not in the domain of f, so that $f(g(2))$ is undefined.

Similarly, the codomain of f is not a subset of the domain of g, so the composition $g \circ f$ is undefined. For example, $f(1) = -2$ which is not in the domain of g, so $g(f(1))$ is undefined.

(d) The codomain of f is not a subset of the domain of g, so that $g \circ f$ is undefined.

On the other hand, $f \circ g : A \to C$ is a well-defined function. It is given by

$$(f \circ g)(a) = f(g(a)) = f(2) = y; \qquad (f \circ g)(b) = f(g(b)) = f(3) = y.$$

That is, $f \circ g$ is the constant y-function from A to C.

(e) This question teaches us to be careful; the "function" f is not well-defined! Indeed, $[0]_5 = [5]_5$, but

$$f([0]_5) = [1]_2 \neq [0]_2 = [6]_2 = f([5]_5).$$

Therefore, neither $f \circ g$ nor $g \circ f$ is a well-defined function (regardless of their domain or codomain).

Helpful Tip!

Knowing this, you should go back to part (b) and verify the functions there are indeed well-defined!

Revisit §8.1 from the recommended reading and/or Exercise 9 from the Introduction to Functions handout if you'd like some more practice with functions defined via representatives.

(Exercise on page 151.)

Solution for Exercise 135 (Order of composition).

(a) One of the simplest examples is $a = b = c = 0$ and $d = 1$; i.e., $f(x) = 0$ is the constant 0 function and $g(x) = 1$ is the constant 1 function. Then, for any $x \in \mathbb{R}$, we have

$$(f \circ g)(x) = f(g(x)) = f(1) = 0;$$
$$(g \circ f)(x) = g(f(x)) = g(0) = 1.$$

(b) Let us compute the compositions $f \circ g$ and $g \circ f$ and see what conditions a, b, c, d must satisfy in order for these two functions to be equal. For any $x \in \mathbb{R}$ we have

$$(f \circ g)(x) = f(g(x)) = f(cx + d) = a(cx + d) + b = acx + ad + b;$$
$$(g \circ f)(x) = g(f(x)) = g(ax + b) = c(ax + b) + d = acx + bc + d.$$

We see that $f \circ g = g \circ f$ if and only if $ad + b = bc + d$.

Helpful Tip!

Note that if $b = d = 0$ the functions commute. This is just a fancy way of restating the fact that multiplication of real numbers is commutative! (Do you see why?)

(c) We claim that only the identity function $h(x) = x$ commutes with all affine functions. It is clear that it does commute: for any $x \in \mathbb{R}$,

$$(f \circ h)(x) = f(h(x)) = f(x) = h(f(x)) = (h \circ f)(x).$$

On the other hand, if h is not the identity function, let $r \in \mathbb{R}$ be such that $h(r) \neq r$. Let f be the constant r-function (i.e. $f(x) = r$). Then, for any $x \in \mathbb{R}$,

$$(f \circ h)(x) = f(h(x)) = r \neq h(r) = h(f(x)) = (h \circ f)(x).$$

(Exercise on page 152.)

Solution for Exercise 136 (Compositions and injectivity).

(a) Suppose $f : X \to Y$ and $g : Y \to Z$ are injective. To prove that $g \circ f : X \to Z$ is injective, suppose $x_1, x_2 \in X$ are such that $(g \circ f)(x_1) = (g \circ f)(x_2)$. Then,

$$\begin{aligned} g(f(x_1)) &= g(f(x_2)) && \text{by the definition of composition;} \\ f(x_1) &= f(x_2) && \text{since } g \text{ is injective;} \\ x_1 &= x_2 && \text{since } f \text{ is injective.} \end{aligned}$$

This proves that $g \circ f$ is injective.

(b) For a simple example (cf. Exercise 2 of the Injective and Surjective Functions handout), suppose $X = [0, \infty) = Z$ and $Y = \mathbb{R}$. Take f to be the inclusion function $f(x) = x$ and g the squaring function $g(x) = x^2$. Then f is injective and g is not injective. On the other hand, $g \circ f : [0, \infty) \to [0, \infty)$ is the squaring function:

$$(g \circ f)(x) = g(f(x)) = g(x) = x^2$$

which is injective (even bijective) on the domain $[0, \infty)$. Indeed, if x_1, x_2 are both positive reals, then $x_1^2 = x_2^2$ if and only if $x_1 = x_2$.

(c) It is not possible for both f, g to be non-injective and for $g \circ f$ to be injective. Indeed, we prove that if $g \circ f$ is injective, then so is f.

Suppose $g \circ f$ is injective; to prove that f is injective, let $x_1, x_2 \in X$ be any elements such that $f(x_1) = f(x_2)$. Then $g(f(x_1) = g(f(x_2))$ (the input to g is the same!). That is, $(g \circ f)(x_1) = (g \circ f)(x_2)$. Since $g \circ f$ is injective, we have $x_1 = x_2$.

This shows that $f(x_1) = f(x_2)$ implies $x_1 = x_2$; i.e. that f is injective.

(d) Since i_X is injective (indeed, bijective), it must be the case that f is injective. This shows that if f has a left-inverse, then f is injective.

In fact, this last assertion is an "if and only if" as we will soon prove.

(Exercise on page 153.)

Solution for Exercise 137 (Compositions and surjectivity)**.**

(a) Suppose $f : X \to Y$ and $g : Y \to Z$ are surjective. To prove that $g \circ f : X \to Z$ is surjective, choose an arbitrary $z \in Z$.

Since g is surjective, there exists some $y \in Y$ with $g(y) = z$.

Since f is surjective, there exists some $x \in X$ with $f(x) = y$.

Then, $(g \circ f)(x) = g(f(x)) = g(y) = z$. Since z was arbitrary, this proves that $g \circ f$ is surjective.

(b) For a very simple example, let $f : \{0\} \to \mathbb{R}$ be the constant 0-function and $g : \mathbb{R} \to \{0\}$ be the constant 0-function. Then $g \circ f : \{0\} \to \{0\}$ is the constant 0-function and is surjective (indeed, bijective).

(c) It is not possible for both f, g to be non-surjective and for $g \circ f$ to be surjective. Indeed, we prove that if $g \circ f$ is surjective, then so is g.

Suppose $g \circ f$ is surjective; to prove that g is surjective, let $z \in Z$ be arbitrary. Since $g \circ f$ is surjective, there must be some $x \in X$ such that $(g \circ f)(x) = z$. Then $g(f(x)) = z$, proving that there is some $y \in Y$ such that $g(y) = z$ (namely, $y = f(x)$). Since z was arbitrary, this proves that g is surjective.

(d) Since i_X is surjective (indeed, bijective), it must be the case that g is surjective. This shows that if g has a right-inverse, then g is surjective.

In fact, this last assertion is an "if and only if" as we will soon prove.

(Exercise on page 154.)

Solution for Exercise 138 (Left- and right-inverses).

(a) Complete the following table

	injective	surjective	has a left-inverse	has a right-inverse
f_1	yes	no	yes	no
f_2	no	yes	no	yes
f_3	yes	yes	yes	yes
f_4	no	no	no	no
f_5	no	no	no	no

(b) The function f_1 has a left-inverse $g_1 : \{a, b, c\} \to \{1, 2\}$ given by $g_1(a) = 1$, $g_1(b) = 2$ and $g_1(c)$ can be chosen arbitrarily (say, $g_1(c) = 1$).

The function f_2 has a right-inverse $g_2 : \{a, b\} \to \{1, 2, 3\}$ given by $g_2(a) = 1$ (it is also possible to define $g_2(a) = 2$) and $g_2(b) = 3$.

The function f_3 has a left-inverse and a right-inverse, and it is the same function $g_3 : \mathbb{R} \to \mathbb{R}$ given by $g(x) = (x - 1)/2$.

(c) Let us address each function in turn.

- The function f_1 is injective but not surjective; we can modify the codomain to be $\{a, b\}$. This would make the resulting function bijective and therefore it would have both a left- and a right- inverse.
- The function f_2 is surjective but not injective; we can modify the domain to be $\{1, 3\}$ or $\{2, 3\}$. This would make the resulting function bijective and therefore it would have both a left- and a right- inverse.
- The function f_4 is neither injective nor surjective. Changing the domain to $[0, +\infty)$ (or to $(-\infty, 0]$; or many other possibilities) would make the resulting function injective and would therefore guarantee the existence of a left-inverse.

  Changing the codomain to $[0, +\infty)$ would make the resulting function surjective and would guarantee the existence of a right-inverse.

  Making both changes (say $f_4' : (-\infty, 0] \to [0, \infty)$) would create a bijective function with both a left- and a right-inverse.
- The function f_5 is neither injective nor surjective. Restricting the domain to $[0]_6, [1]_6$ (there are other possibilities) would result in an injective function that would have a left-inverse.

  Changing the codomain to $[0]_6, [3]_6$ would result in a surjective function that would have a right-inverse.

  Making both changes (say $f_5' : \{[0]_6, [1]_6\} \to \{[0]_6, [3]_6\}$) would result in a bijective function with both a left- and right-inverse.

(Exercise on page 155.)

Solution for Exercise 139 (Inverse relation).

(a) Note that since $R^{-1} \subseteq B \times A$ we have $(R^{-1})^{-1} \subseteq A \times B$.

Let $(a, b) \in A \times B$ be arbitrary. Then,

$$\begin{aligned}(a,b) \in (R^{-1})^{-1} &\iff (b,a) \in R^{-1} && \text{by the definition of the inverse of } R^{-1};\\ &\iff (a,b) \in R && \text{by the definition of the inverse of } R.\end{aligned}$$

This shows that R and $(R^{-1})^{-1}$ have precisely the same elements; i.e., they are the same subset of $A \times B$.

(b) Fix some $b \in B$ such that $\forall a \in A.(b, a) \notin R^{-1}$. Since $(b, a) \in R^{-1} \iff (a, b) \in R$, we see that $\forall a \in A.(a, b) \notin R$. This means that b is not in the image of the function R; in other words, R cannot be surjective.

(c) We claim that every $b \in B$ has at least one $a \in A$ for which $(b, a) \in R^{-1}$ if and only if R is surjective.

We have already shown above (via the contrapositive) that if R is surjective then every $b \in B$ has at least one $a \in A$ for which $(b, a) \in R^{-1}$.

Conversely, if every $b \in B$ has at least one $a \in A$ for which $(b, a) \in R^{-1}$ then every $b \in B$ has at least one $a \in A$ for which $(a, b) \in R$ (since $(b, a) \in R^{-1} \iff (a, b) \in R$), proving that R is surjective.

(d) Suppose $b \in B$ and $a \neq a' \in A$ are such that $(b, a); (b, a') \in R^{-1}$. Then $(a, b); (a', b) \in R$ which means that $R(a) = R(a')$ even though $a \neq a'$; in other words, R cannot be injective.

(e) We claim that every $b \in B$ has at most one $a \in A$ for which $(b, a) \in R^{-1}$ if and only if R is injective.

We have already shown above (via the contrapositive) that if R is injective then every $b \in B$ has at most one $a \in A$ for which $(b, a) \in R^{-1}$.

Conversely, suppose every $b \in B$ has at most one $a \in A$ for which $(a, b) \in R^{-1}$. To prove that R is injective, let $a, a' \in A$ be such that $R(a) = b = R(a')$. Then $(a, b); (a', b) \in R$ and therefore $(b, a); (b, a') \in R^{-1}$. By the condition on R^{-1} we must have $a = a'$.

(f) We claim that R^{-1} is a function if and only if R is a bijection.

In one direction, suppose R^{-1} is a function. Then every $b \in B$ has exactly one $a \in A$ such that $(b, a) \in R^{-1}$ so that by parts (c) and (e) above, it must be that R is surjective and injective, hence bijective.

Conversely, suppose R is bijective. Then R is surjective and injective and so by parts (c) and (e) above we know that for every $b \in B$ there is exactly one $a \in A$ such that $(b, a) \in R^{-1}$, i.e. R^{-1} is a function.

(Exercise on page 156.)

Solution for Exercise 140 (Two-sided inverse)**.**

(a) A left-inverse for f_1 is a function $h : \{a, b, c\} \to \{1, 2\}$ such that $h \circ f_1$ is the identity on $\{1, 2\}$. In particular, we must have $1 = h(f_1(1)) = h(a)$ and $2 = h(f_1(2)) = h(b)$. The value of c is free. Therefore,

$$\begin{aligned} &g_1 : \{a, b, c\} \to \{1, 2\} &&\text{given by } g_1(a) = 1, g_1(b) = 2, g_1(c) = 1; \\ &g_1' : \{a, b, c\} \to \{1, 2\} &&\text{given by } g_1'(a) = 1, g_1'(b) = 2, g_1'(c) = 2 \end{aligned}$$

are two distinct left-inverses for f_1.

(b) A right-inverse for f_2 is a function $h : \{a, b\} \to \{1, 2, 3\}$ such that $f_2 \circ h$ is the identity on $\{a, b\}$. In particular, we must have $a = f_2(h(a))$ and $b = f_2(h(b))$. Since $f_2(x) = b$ if and only if $x = 3$, we must have $h(b) = 3$. On the other hand, we have two choices for the value of $h(a)$. Therefore,

$$\begin{aligned} &g_2 : \{a, b\} \to \{1, 2, 3\} &&\text{given by } g_2(a) = 1, g_2(b) = 3; \\ &g_2' : \{a, b\} \to \{1, 2, 3\} &&\text{given by } g_2'(a) = 2, g_2'(b) = 3 \end{aligned}$$

are two distinct right-inverses for f_2.

(c) Suppose $f : X \to Y$ is a function which has a left-inverse $g : Y \to X$ and a right-inverse $h : Y \to X$. Since g, h have the same domain and codomain, it remains to prove that they have the same rule; i.e. $g(y) = h(y)$ for every $y \in Y$.

Let $y \in Y$ be arbitrary. Consider the value of $(g \circ f \circ h)(y)$. Since function composition is associative, we have

$$(g \circ f \circ h)(y) = g((f \circ h)(y)) = g(i_Y(y)) = g(y),$$

where we have used the fact that $f \circ h = i_Y$ since h is a right-inverse for f. On the other hand,

$$(g \circ f \circ h)(y) = (g \circ f)(h(y)) = i_X(h(y)) = h(y),$$

where we have used the fact that $g \circ f = i_X$ since g is a left-inverse for f. In conclusion,

$$g(y) = (g \circ f \circ h)(y) = h(y).$$

(d) Suppose f has both a left-inverse and a right-inverse. Fix any left-inverse and call it g. We have shown in the previous part that *any* right-inverse h for f is equal to g; in particular, there is a *unique* right-inverse.

The same argument with the roles of g and h interchanged show there is a *unique* left-inverse. Finally, we have already shown that the left-inverse and the right-inverse must equal each other, creating a two-sided inverse. Moreover, this two-sided inverse is unique (because any two-sided inverse is in particular a right-inverse, say).

(e) The previous part shows that whenever a two-sided inverse exists, it is unique. Therefore, suffice it to show that $f^{-1} \circ g^{-1}$ is a two-sided inverse for $g \circ f : X \to Z$. Using the associativity of composition,

$$\begin{aligned} (f^{-1} \circ g^{-1}) \circ (g \circ f) &= f^{-1} \circ ((g^{-1} \circ g) \circ f) \\ &= f^{-1}(\circ(i_Y) \circ f) \\ &= f^{-1} \circ f \\ &= i_X \end{aligned}$$

proving that $f^{-1} \circ g^{-1}$ is a left-inverse for $g \circ f$. Similarly,

$$\begin{aligned}(g \circ f) \circ (f^{-1} \circ g^{-1}) &= (g \circ (f \circ f^{-1})) \circ g^{-1} \\ &= (g \circ i_X) \circ g^{-1} \\ &= g \circ g^{-1} \\ &= i_Z\end{aligned}$$

proving that $f^{-1} \circ g^{-1}$ is a right-inverse for $g \circ f$.

(Exercise on page 157.)

Solution for Exercise 141 (Cantor–Schröder–Bernstein Theorem).

(a) Suppose towards contradiction that $\phi^{-1}(\{b\})$ contains at least two elements, say $a_1 \neq a_2$. Then $\phi(a_1) = b = \phi(a_2)$, contradicting the assumption that ϕ is injective.

(b) We show the proof for ϕ_n, the proof for ψ_n is completely analogous.

We prove that for even n, ϕ_n is an injective function $Y \to X$; whereas for odd n, ϕ_n is an injective function $X \to X$.

Indeed, $\phi_0 = g$ is an injective function $Y \to X$. Suppose that for some $n \in \mathbb{N}$ we know that ϕ_n is a well-defined injective function with domain Y and codomain X if n is even and domain X and codomain X if n is odd.

- Suppose n is even, so that $\phi_n : Y \to X$ is a well-defined injective function. Since $n+1$ is odd, $\phi_{n+1} = \phi_n \circ f$. Now, $f : X \to Y$ and $\phi_n : Y \to X$, so that $\phi_n \circ f$ is a well-defined function $X \to X$. Moreover, it is injective as the composition of two injective functions (cf. Exercise 136).
- Suppose n is odd, so that $\phi_n : X \to X$ is a well-defined injective function. Since $n+1$ is even, $\phi_{n+1} = \phi_n \circ g$. Now, $g : Y \to X$ and $\phi_n : X \to X$, so that $\phi_n \circ g$ is a well-defined function $Y \to X$. Moreover, it is injective as the composition of two injective functions.

We conclude that ϕ_{n+1} is a well-defined injective function with domain Y and codomain X if $n+1$ is even, and domain X and codomain X if $n+1$ is odd. This completes the inductive proof.

(c) Since ϕ_n is injective for every n, we know by part (a) that $\phi_n^{-1}(\{x\})$ has at most 1 element. The analogous statement holds for $\psi_n^{-1}(\{y\})$.

(d) We prove by mathematical induction that for every $n \in \mathbb{N}$ we have $f \circ \phi_{n-1} = \psi_n$; the proof that $g \circ \psi_{n-1} = \phi_n$ is analogous.

Note that
$$\psi_1 = \psi_0 \circ g = f \circ g = f \circ \phi_0.$$

Suppose that for some $n \in \mathbb{N}$ we have already shown that $f \circ \phi_{n-1} = \psi_n$.

- Suppose n is odd, then
$$\begin{aligned}
\psi_{n+1} &= \psi_n \circ f && \text{by the definition of } \psi_{n+1}\text{, since } n+1 \text{ is even;}\\
&= (f \circ \phi_{n-1}) \circ f && \text{by the induction hypothesis;}\\
&= f \circ (\phi_{n-1} \circ f) && \text{by associativity of composition;}\\
&= f \circ \phi_n && \text{by the definition of } \phi_n\text{, since } n \text{ is odd.}
\end{aligned}$$
- Suppose n is even, then
$$\begin{aligned}
\psi_{n+1} &= \psi_n \circ g && \text{by the definition of } \phi_{n+1}\text{, since } n+1 \text{ is odd;}\\
&= (f \circ \phi_{n-1}) \circ g && \text{by the induction hypothesis;}\\
&= f \circ (\phi_{n-1} \circ g) && \text{by associativity of composition;}\\
&= f \circ \phi_n && \text{by the definition of } \phi_n\text{, since } n \text{ is even.}
\end{aligned}$$

Either way, we see that $f \circ \phi_n = \psi_{n+1}$. This complete the inductive proof.

(e) Let $x \in X$ and $n \in \mathbb{N}$ be arbitrary.

We start by proving that $\phi_{n-1}^{-1}(\{x\}) \subseteq \psi_n^{-1}(\{f(x)\})$. (This is obvious if $\phi_{n-1}^{-1}(\{x\}) = \emptyset$.) Towards that end, let $z \in \phi_{n-1}^{-1}(\{x\})$ so that $\phi_{n-1}(z) = x$. Applying f to both sides we have $(f \circ \phi_{n-1})(z) = f(x)$. By the previous part, $f \circ \phi_{n-1} = \psi_n$, so that $\psi_n(z) = f(x)$, showing that $z \in \psi_n^{-1}(\{f(x)\})$.

Similarly, we prove that $\psi_n^{-1}(\{f(x)\}) \subseteq \phi_{n-1}^{-1}(\{x\})$. (This is obvious if $\psi_n^{-1}(\{f(x)\}) = \emptyset$.) Towards that end, let $z \in \psi_n^{-1}(\{f(x)\})$ so that $\psi_n(z) = f(x)$. Applying the previous part, $\psi_n = f \circ \phi_{n-1}$ so that $f(\phi_{n-1}(z)) = f(x)$. Since f is injective, we conclude that $\phi_{n-1}(z) = x$ so that $z \in \phi_{n-1}^{-1}(\{x\})$.

Helpful Tip!

The claim in this part more easily follows from the fact that for any relations R, S for which $R \circ S$ is defined, we always have $(R \circ S)^{-1} = S^{-1} \circ R^{-1}$.

The analogous statement for $y \in Y$ is:

$$\forall y \in Y\ \forall n \in \mathbb{N}\ (\phi_n^{-1}(\{g(y)\}) = \psi_{n-1}^{-1}(\{y\})).$$

(f) We start by proving that $f(X_{no}) \subseteq Y_{no}$. For any $x \in X_{no}$ and any $n \in \mathbb{N}$, we have $\phi_{n-1}^{-1}(\{x\}) \neq \emptyset$ and therefore (applying the previous part) $\psi_n^{-1}(\{f(x)\}) \neq \emptyset$. Moreover, since $\psi_0 = f$, we clearly have $\psi_0^{-1}(\{f(x\}) \neq \emptyset$ (namely, it has x as an element). This proves that $\forall n \in \mathbb{Z}_{\geq 0}\ \psi_n^{-1}(\{f(x)\}) \neq \emptyset$, i.e. that $f(x) \in Y_{no}$.

Next, we show that $Y_{no} \subseteq f(X_{no})$. Towards that end, let $y \in Y_{no}$ be an arbitrary element. By the definition of Y_{no}, we must have $\psi_0^{-1}(\{y\}) \neq \emptyset$. Since $\psi_0 = f$, we conclude that there is some $x \in X$ for which $f(x) = y$. Applying the previous part, for any $n \in \mathbb{N}$, we have $\phi_{n-1}^{-1}(\{x\}) \neq \psi_n^{-1}(\{f(x)\}) \neq \emptyset$ (because $f(x) = y \in Y_{no}$). This proves that $x \in X_{no}$.

We have now shown that $f(X_{no}) = Y_{no}$, i.e. that $f' : X_{no} \to Y_{no}$ given by $f'(x) = f(x)$ is surjective. Since f is injective, so must be f'; so that f' is indeed a bijection.

(g) We start by showing that $f(X_{even}) \subseteq Y_{odd}$. Let $x \in X_{even}$ be an arbitrary element. This means the smallest $n \in \mathbb{Z}_{\geq 0}$ such that $\phi_n^{-1}(\{x\}) = \emptyset$ is even, and we denote this even number by m. Consider now $y = f(x)$. Note that $\psi_0^{-1}(\{y\}) = \{x\} \neq \emptyset$, so that the smallest $n \in \mathbb{Z}_{\geq 0}$ (if it exists) is greater than 1. Applying part (e), we note that for every $n \in \mathbb{N}$,

$$\phi_{n-1}^{-1}(\{x\}) = \psi_n^{-1}(\{f(x)\})$$

so the smallest $n \in \mathbb{Z}_{\geq 0}$ such that $\psi_n^{-1}(\{y\}) = \emptyset$ is simply $m + 1$. Since $m + 1$ is odd (as m is even), this proves that $f(x) \in Y_{odd}$.

Next, we show that $Y_{odd} \subseteq f(X_{even})$. Towards that end, let $y \in Y_{odd}$ be an arbitrary element. In particular, the smallest $n \in \mathbb{Z}_{\geq 0}$ such that $\psi_n^{-1}(\{y\}) = \emptyset$ (which we shall call k) is at least 1, so that $\psi_0^{-1}(\{y\}) \neq \emptyset$. Therefore (since $\psi_0 = f$), there is some $x \in X$ such that $f(x) = y$. Applying part (e), for every $n \in \mathbb{N}$,

$$\phi_{n-1}^{-1}(\{x\}) = \psi_n^{-1}(\{f(x)\})$$

so that the smallest $t \in \mathbb{Z}_{\geq 0}$ for which $\phi_t^{-1}(\{x\}) = \emptyset$ is simply $k - 1$, which is even. This proves that $x \in X_{even}$.

We have now shown that $f(X_{even}) = Y_{odd}$, i.e. that $f'' : X_{even} \to Y_{odd}$ given by $f''(x) = f(x)$ is surjective. Since f is injective, so must be f''; so that f'' is indeed a bijection.

(h) The proof that $g(Y_{even}) = X_{odd}$ is entirely analogous to the previous part (interchange x and y and interchange ϕ and ψ). It follows that $g'' : Y_{even} \to X_{odd}$ (given by $g''(y) = g(y)$) is a bijection. Therefore, it has a well-defined inverse $(g'')^{-1} : X_{odd} \to X_{even}$ which is also bijective[31]. This is the function we called g' in the exercise statement.

(i) We define

$$h(x) = \begin{cases} f(x) & \text{if } x \in X_{no} \cup X_{even}; \\ g^{-1}(x) & \text{if } x \in X_{odd}. \end{cases}$$

Parts (d) and (e) above show that the restriction of f to $X_{no} \cup X_{even}$ defines a bijection to $Y_{no} \cup Y_{odd}$, while part (f) shows that the restriction of g^{-1} to X_{odd} defines a bijection to Y_{even}. Since $X_{no}, X_{even}, X_{odd}$ partition X and $Y_{no}, Y_{even}, Y_{odd}$ partition Y, we conclude that h is a well-defined bijection from X to Y.

(Exercise on page 158.)

[31]This is Theorem 8.77 of the recommended reading; but it also follows from Exercise 139 because $((g'')^{-1})^{-1} = g''$ is a function, so the function $(g'')^{-1}$ must be bijective.

Solution for Exercise 142 (Notation)**.**

(a) $f^{-1}(3)$ is undefined; because f^{-1} is applied to element, it refers to the inverse function. Since f is neither one-to-one nor onto, it is not invertible.

(b) $f^{-1}(\{3\}) = \{-\sqrt{3}, \sqrt{3}\}$. In this case, f^{-1} is applied to a set and therefore refers to the preimage of that set. The preimage is all the possible inputs x such that $f(x) = x^2 = 3$.

> **Helpful Tip!**
>
> Note that if f^{-1} is applied to an element, the result should be an element; if f^{-1} is applied to a set, the result should be a set!
>
> Sometimes when f does not have a well-defined inverse, one sees $f^{-1}(a)$ to mean $f^{-1}(\{a\})$, because the meaning of the notation is unambiguous. We strongly advise you to avoid such notation.

(c) $f^{-1}(\{-3\}) = \emptyset$. Once again, the notation refers to the preimage; all the possible inputs x such that $f(x) = x^2 = -3$. This set happens to be empty!

(d) $f^{-1}(x)$ is undefined. In this case, f^{-1} refers to the inverse function, which does not exist.

(e) $f^{-1}([0, 1]) = [-1, 1]$. The input is a set, so $f^{-1}([0, 1])$ is the preimage of the elements in the set $[0, 1]$.

(Exercise on page 159.)

Solution for Exercise 143 (Images and Preimages).

(a) We have $S = \{-1, 0, 2, 4, 7\}$; let us find the image $f(S)$ for each function f in turn:

(i) The constant function $f(x) = 1$. All inputs are mapped to 1, so $f(S) = \{1\}$.

(ii) The linear function $f(x) = 2x + 1$. Applying the function rule to each element in S, we have $f(S) = \{-1, 1, 5, 9, 15\}$. For example,

$$f(-1) = 2(-1) + 1 = -1, \qquad f(7) = 2(7) + 1 = 15.$$

(iii) The inclusion function $\iota : S \to \mathbb{Z}$. Since every element maps to itself, $f(S) = S$.

(iv) The function $f(x) = \lceil \frac{x}{5} \rceil$. Applying the function rule to each element in S, we have $f(S) = \{0, 1, 2\}$. For example,

$$f(-1) = \left\lceil \frac{-1}{5} \right\rceil = 0, \qquad f(7) = \left\lceil \frac{7}{5} \right\rceil = 2.$$

(b) We have $f(x) = 2x$; let us compute the image $f(S)$ for each S in turn.

(i) $S = \{-2, -1, 0, \frac{1}{2}, \frac{5}{6}, \pi\}$. Doubling each element, we have $f(S) = \{-4, -2, 0, 1, \frac{5}{3}, 2\pi\}$.

(ii) $S = \mathbb{N}$. Doubling each element, we find

$$f(\mathbb{N}) = \{f(x) : x \in \mathbb{N}\} = \{2x : x \in \mathbb{N}\} = \text{the set of even natural numbers.}$$

(iii) $S = \mathbb{Z}$. Doubling each element, we find

$$f(\mathbb{Z}) = \{f(x) : x \in \mathbb{Z}\} = \{2x : x \in \mathbb{Z}\} = \text{the set of even integers.}$$

(iv) $S = \mathbb{R}$. We claim that $f(\mathbb{R}) = \mathbb{R}$, which is the same as saying that f is surjective. Indeed, given an arbitrary $y \in \mathbb{R}$ we have $f(y/2) = y$.

(c) We have $f(x) = |x|$, let us find the preimage of each set in turn.

(i) The preimage of the singleton set $\{4\}$ is the set of inputs that map to 4, i.e., all x such that $|x| = 4$. Therefore, $f^{-1}(\{4\}) = \{-4, 4\}$.

(ii) Each element y in the interval $[2, 8]$ is positive and has exactly two preimages, namely y and $-y$. Therefore, $f^{-1}([2, 8]) = [2, 8] \cup [-8, -2]$

(iii) Each positive integer n has exactly two preimages, namely n and $-n$. Every negative integer has an empty preimage, because negative integers are not in the range of f. Finally, the integer 0 has exactly one preimage, namely 0 itself. Therefore, $f^{-1}(\mathbb{Z}) = \mathbb{Z}_{\geq 0}$.

(iv) Every negative real number has an empty preimage, because negative real numbers are not in the range of f. And the integer 0 has exactly one preimage, namely 0 itself. Therefore, $f^{-1}((-\infty, 0]) = \{0\}$.

(Exercise on page 160.)

Solution for Exercise 144 (Preimages and Complements).

(a) We have

$$f^{-1}(S) = \{x \in X : f(x) \in S\}$$
$$S^c = \{y \in Y : y \notin S\}.$$

(b) We prove that $f^{-1}(S^c) = (f^{-1}(S))^c$ by showing that each set is contained in the other.

To show that $f^{-1}(S^c) \subseteq (f^{-1}(S))^c$, let $x \in f^{-1}(S^c)$ be arbitrary. By the definition of preimage, this means that $f(x) \in S^c$. By the definition of complement, we have $f(x) \notin S$. Using the definition of preimage again, we have $x \notin f^{-1(S)}$, and applying the definition of complement again, that $x \in (f^{-1}(S))^c$.

To show that $(f^{-1}(S))^c \subseteq f^{-1}(S^c)$, let $x \in (f^{-1}(S))^c$ be arbitrary. Then, $x \notin f^{-1}(S)$ so that $f(x) \notin S$. Hence, $f(x) \in S^c$ and therefore $x \in f^{-1}(S^c)$.

(c) In words, this equality says that if you *first* take a complement in the codomain and *then* take the preimage under f, you get the same set as if you *first* take the preimage under f and *then* take the complement in the domain.

There are several popular ways to express this pithily:

- "preimages 'respects' complements";[32]
- "preimages 'commute' with complements";
- "the preimage of the complement is the complement of the preimage."

(Exercise on page 161.)

[32]This is a shorthand for "[The operation of taking] preimages 'respects' [the operation of taking] complements". Similar comment applies to the next item.

Solution for Exercise 145 (Images and Intersections).

(a) To show that $f(A \cap B) \subseteq f(A) \cap f(B)$, let $y \in f(A \cap B)$ be an arbitrary element.

By the definition of image, there exists some $x \in A \cap B$ such that $f(x) = y$. Since $x \in A \cap B$, we have $x \in A$ and $x \in B$. Therefore, $y = f(x) \in f(A)$ and $y = f(x) \in f(B)$, so $y \in f(A) \cap f(B)$.

(b) For an example where equality holds, let $f : \mathbb{R} \to \mathbb{R}$ be the identity function. Then for any $Z \subseteq \mathbb{R}$ we have $f(Z) = Z$. In particular,

$$f(A \cap B) = A \cap B = f(A) \cap f(B).$$

For an example where equality fails, let $f : \mathbb{R} \to \mathbb{R}$ be the constant function $f(x) = 1$. Then for any nonempty $\emptyset \neq Z \subseteq \mathbb{R}$ we have $f(Z) = \{1\}$. Therefore, if A, B are disjoint (say $A = \{0\}$ and $B = \{1\}$) we have

$$f(A \cap B) = f(\emptyset) = \emptyset \subsetneq \{1\} = \{1\} \cap \{1\} = f(A) \cap f(B).$$

Another example is the squaring function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$, and the sets $A = [-2, 0]$, and $B = [0, 2]$. Then $A \cap B = \{0\}$, so

$$f(A \cap B) = f(\{0\}) = \{0\}.$$

On the other hand,

$$f(A) = f([-2, 0]) = [0, 4], \qquad f(B) = f([0, 2]) = [0, 4],$$

so

$$f(A) \cap f(B) = [0, 4].$$

Therefore $f(A \cap B) \subsetneq f(A) \cap f(B)$.

(c) Suffice it to guarantee that $f(A) \cap f(B) \subseteq f(A \cap B)$. Suppose $y \in f(A) \cap f(B)$; this means that $y \in f(A)$ and $y \in f(B)$. Therefore, there is some $a \in A$ such that $f(a) = y$, and also some $b \in B$ such that $f(b) = y$.

Since we want to show that $y \in f(A \cap B)$, we would like to find some $x \in A \cap B$ such that $f(x) = y$. It would be nice to take $x = a$ or $x = b$, but we cannot guarantee that these elements are in the intersection $A \cap B$. One way to guarantee this is if $a = b$. Unpacking this discussion, we see that if f is injective (one-to-one), then

$$f(A \cap B) = f(A) \cap f(B) \quad \text{for all } A, B \subseteq X.$$

Let's prove this formally. Assume f is injective, and let $y \in f(A) \cap f(B)$ be arbitrary. Then $y \in f(A)$ and $y \in f(B)$, so there exist $a \in A$ and $b \in B$ such that $f(a) = y$ and $f(b) = y$. In particular, $f(a) = y = f(b)$, so injectivity implies $a = b$, and we call this common element x. Then $a = x \in A$ and $b = x \in B$, so $x \in A \cap B$. Moreover, $f(x) = y$, so $y \in f(A \cap B)$.

(d) We prove that if $f(A \cap B) = f(A) \cap f(B)$ for all $A, B \subseteq X$, then f is injective.

Suppose $x_1, x_2 \in X$ are such that $f(x_1) = f(x_2)$, call this common value y. We prove that $x_1 = x_2$. Indeed, let $A = \{x_1\}$ and $B = \{x_2\}$. Then

$$f(A) \cap f(B) = \{f(x_1)\} \cap \{f(x_2)\} = \{y\}.$$

Then our hypothesis gives

$$f(A \cap B) = f(A) \cap f(B) = \{y\}.$$

In particular, there must be some $x \in A \cap B$ such that $f(x) = y$; so that $A \cap B$ is nonempty. But $A \cap B \subseteq A = \{x_1\}$, so if $A \cap B$ is nonempty, we must have $A \cap B = \{x_1\}$. Similarly, $A \cap B \subseteq B = \{x_2\}$, so we must have $A \cap B = \{x_2\}$. Therefore, $\{x_1\} = A \cap B = \{x_2\}$, proving that $x_1 = x_2$.

Helpful Tip!

Compare to Exercise 6 in the handout on Injective and Surjective Functions. The similarity can be explained by observing that Exercise 6 is actually a consequence of the current exercise together with Exercise 144, since $U \setminus V = U \cap V^c$.

(Exercise on page 162.)

Solution for Exercise 146 (The Characterisitc Function)**.**

(a) If $S = \emptyset$, then no element of U lies in S, so $\chi_S(x) = 0$ for all $x \in U$. Hence, $\chi_\emptyset(U) = \{0\}$.

If $S = U$, then every element of U lies in S, so $\chi_S(x) = 1$ for all $x \in U$. Hence, $\chi_U(U) = \{1\}$.

If S is a proper, nonempty subset of U, then there exists at least one $x \in S$ (because S is nonempty) and at least one $y \in U \setminus S$ (because S is proper). Then, $\chi_S(x) = 1$ and $\chi_S(y) = 0$, so both 0 and 1 occur as values. Hence, $\chi_S(U) = \{0, 1\}$.

(b) We have

$$\begin{aligned}
\chi_S^{-1}(\{1\}) &= \{x \in U \mid \chi_S(x) = 1\} = \{x \in U \mid x \in S\} = S, \\
\chi_S^{-1}(\{0\}) &= \{x \in U \mid \chi_S(x) = 0\} = \{x \in U \mid x \notin S\} = S^c, \\
\chi_S^{-1}(\{0,1\}) &= \{x \in U \mid \chi_S(x) \in \{0,1\}\} = U \quad \text{(since } \chi_S \text{ always takes values in } \{0,1\}), \\
\chi_S^{-1}(\emptyset) &= \{x \in U \mid \chi_S(x) \in \emptyset\} = \emptyset.
\end{aligned}$$

So the preimages recover S, its complement S^c, the universal set U, and the empty set $\emptyset$.

(c) This immediately follows from the fact that $0^2 = 0$ and $1^2 = 1$. Therefore, for any $x \in U$ we have $\chi_S(x)^2 = \chi_S(x)$. Since χ_S^2 and χ_S have the same domain, codomain, and rule, they must be the same function!

(d) For any $x \in U$, note that $\chi_A(x) \cdot \chi_B(x) \in \{0, 1\}$ and is only 1 if both $\chi_A(x) = 1 = \chi_B(x)$, i.e. if and only if $x \in A$ and $x \in B$. Therefore,

$$\chi_A(x) \cdot \chi_B(x) = \begin{cases} 1 & \text{if } x \in A \cap B; \\ 0 & \text{otherwise.} \end{cases}$$

This is precisely the definition of $\chi_{A\cap B}(x)$, so (since both functions have the same domain U, the same codomain $\{0, 1\}$, and the same rule) we have $\chi_A(x) \cdot \chi_B(x) = \chi_{A\cap B}$.

(e) For any $x \in U$, we have

$$\chi_A(x) + \chi_B(x) = \begin{cases} 0 & \text{if } x \text{ is in neither one of } A, B; \\ 1 & \text{if } x \text{ is in exactly one of } A, B; \\ 2 & \text{if } x \text{ is in both of } A, B. \end{cases}$$

In contrast, we have seen above that $\chi_A(x) \cdot \chi_B(x) = 1$ if and only if x is in both A, B. Combining this with the cases above we therefore have

$$\chi_A(x) + \chi_B(x) - \chi_A(x) \cdot \chi_B(x) = \begin{cases} 0 & \text{if } x \text{ is in neither one of } A, B; \\ 1 & \text{if } x \text{ is in exactly one of } A, B; \\ 1 & \text{if } x \text{ is in both of } A, B. \end{cases}$$

We can combine the last two cases:

$$\chi_A(x) + \chi_B(x) - \chi_A(x) \cdot \chi_B(x) = \begin{cases} 0 & \text{if } x \text{ is in neither one of } A, B; \\ 1 & \text{if } x \text{ is in at least one one of } A, B. \end{cases}$$

Finally, using set notation to rewrite the conditions:

$$\chi_A(x) + \chi_B(x) - \chi_A(x) \cdot \chi_B(x) = \begin{cases} 0 & \text{if } x \notin A \cup B; \\ 1 & \text{if } x \in A \cup B. \end{cases}$$

This is precisely the definition of $\chi_{A\cup B}(x)$, so (since we have a match of domain, codomain, and rule) the two functions are indeed the same.

(f) For any $x \in U$ we have

$$1 - \chi_A(x) = \begin{cases} 1-0 & \text{if } x \notin A; \\ 1-1 & \text{if } x \in A. \end{cases}$$

Since $x \notin A \iff x \in A^c$ (and $x \in A \iff x \notin A^c$), we can rewrite this as

$$1 - \chi_A(x) = \begin{cases} 1 & \text{if } x \in A^c; \\ 0 & \text{if } x \notin A^c. \end{cases}$$

This is precisely the definition of χ_{A^c}.

(g) Recall that one expression of the symmetric difference is[33]

$$A \triangle B = (A \cup B) \setminus (A \cap B) = (A \cup B) \cap (A \cap B)^c.$$

Let us denote $S := A \cup B$ and $T = A \cap B$. Then, by part (e) above,

$$\chi_{A \triangle B} = \chi_{S \cap T^c} = \chi_S \chi_{T^c}.$$

Next, by part (e) above,

$$\chi_S = \chi_{A \cup B} = \chi_A + \chi_B - \chi_A \chi_B$$

By part (f) above,

$$\chi_{T^c} = 1 - \chi_T$$

and applying part (d) again,

$$\chi_T = \chi_A \chi_B.$$

Putting everything together, expanding, and using part (c), we find

$$\begin{aligned} \chi_{A \triangle B} &= (\chi_A + \chi_B - \chi_A \chi_B)(1 - \chi_A \chi_B) \\ &= \chi_A + \chi_B - \chi_A \chi_B - \chi_A^2 \chi_B - \chi_A \chi_B^2 + \chi_A^2 \chi_B^2 \\ &= \chi_A + \chi_B - 2\chi_A \chi_B. \end{aligned}$$

Helpful Tip!

We can verify our formula by checking the four possible value of $(\chi_A(x), \chi_B(x))$, namely $(0,0); (1,0); (0,1); (1,1)$. This is also an alternative way of deriving the formula. There are many possible ways to arrive at the formula, we wanted to highlight the algebraic method, which is the "point of" characteristic functions.

Characteristic functions play a crucial role in Mathematical Analysis, and Probability Theory, where they are used in the construction of the Lebesgue integral from the Lebesgue measure (or to assign probability to more general sets).

The reader may enjoy comparing this exercise to Exercise 4 from the handout on Propositional Logic.

(Exercise on page 163.)

[33] There are many equivalent formulations. For example, we could continue from the formulation below using DeMorgan's Laws. However, we choose this one because it allows us to apply the previous parts of the questions in a straightforward manner.

Solution for Exercise 147 (The Characteristic Function of $\mathbb{Z}$).

(a) If $x \in \mathbb{Z}$ then

- The largest integer $\le x$ is x itself, so that $\lfloor x \rfloor = x$;
- The smallest integer $\ge x$ is x itself, so that $\lceil x \rceil = x$.

(b) By their very definition we have $\lfloor x \rfloor \le x \le \lceil x \rceil$, which immediately gives $\lfloor x \rfloor \le \lceil x \rceil$.

(c) We claim that $\lfloor x \rfloor = \lceil x \rceil$ if and only if $x \in Z$. The "if" is exactly part (a) above.

For the "only if", suppose $\lfloor x \rfloor = \lceil x \rceil$. By part (b), this common value must be x itself. In particular, since $\lfloor x \rfloor = x$, it has to be that x is an integer (since, by definition, $\lfloor x \rfloor$ is an integer).

(d) Suppose $\lfloor x \rfloor \ne \lceil x \rceil$. By part (b), this means that $\lfloor x \rfloor < \lceil x \rceil$ and since both of these numbers are integers, $\lfloor x \rfloor + 1 \le \lceil x \rceil$. It remains to prove the reverse inequality.

Now, $\lfloor x \rfloor$ is the greatest integer less than or equal to x, so that $\lfloor x \rfloor + 1$ (being greater than $\lfloor x \rfloor$) cannot be less than or equal to x, i.e. we must have $x \le \lfloor x \rfloor + 1$.

On the other hand, $\lceil x \rceil$ is the least integer greater than or equal to x. Since $\lfloor x \rfloor + 1$ is an integer greater than or equal to x, we must have (by the "least" requirement) $\lceil x \rceil \le \lfloor x \rfloor + 1$.

Therefore, we conclude that $\lfloor x \rfloor + 1 = \lceil x \rceil$.

(e) By part (c) observe that $x \in \mathbb{Z} \iff \lceil x \rceil - \lfloor x \rfloor = 0$, whereas by part (d) $x \notin \mathbb{Z} \iff \lceil x \rceil - \lfloor x \rfloor = 1$. We see that

$$1 - (\lceil x \rceil - \lfloor x \rfloor) = \begin{cases} 1 & \text{if } x \in \mathbb{Z}; \\ 0 & \text{if } x \notin \mathbb{Z}. \end{cases}$$

That is, $\chi_{\mathbb{Z}} = \lfloor x \rfloor - \lceil x \rceil + 1$.

Helpful Tip!

There are other possible solutions! If you'd like more practice with the ceiling and floor functions, show that $\chi_{\mathbb{Z}}(x) = \lfloor x \rfloor + \lfloor -x \rfloor + 1$. (There are more possible expressions.)

(Exercise on page 164.)

Solution for Exercise 148 (Functions, Preimages, and Partitions).

(a) Fix some $y_1 \neq y_2 \in Y$ and suppose towards contradiction $f^{-1}(\{y_1\})$ and $f^{-1}(y_2)$ are not disjoint, i.e. that there exists some $x \in f^{-1}(\{y_1\}) \cap f^{-1}(\{y_2\})$.

Since $x \in f^{-1}(\{y_1\})$, we have $f(x) = y_1$, and since $x \in f^{-1}(\{y_2\})$, we have $f(x) = y_2$. That is, $f(x) = y_1 \neq y_2 = f(x)$, contradicting the fact that f is a function (so each input must have exactly one output). This contradiction proves that the preimages of distinct values are disjoint.

(b) Since $f^{-1}(\{y\}) \subseteq X$, we must have
$$\bigcup_{y \in Y} f^{-1}(\{y\} \subseteq X.$$
For the reverse inclusion, let $x \in X$ be arbitrary and note that $x \in f^{-1}(\{y\})$ for $y = f(x)$. This proves that
$$X \subseteq \bigcup_{y \in Y} f^{-1}(\{y\}.$$
Taken together, these two inclusions prove the two sets are equal.

(c) In general, the collection of preimages may *not* form a partition. This is because the blocks of the partition must be nonempty and there may be $y \in Y$ such that $f^{-1}(\{y\}) = \emptyset$.

There are two simple ways of overcoming this difficulty. First, we can impose conditions on the function. If the function is *surjective* then $\{f^{-1}(\{y\}) : y \in Y\}$ forms a partition of the domain X.

Better (because it works for all function) would be adjust the index set: the collection $\{f^{-1}(\{y\}) : y \in \mathrm{Im}(f$ forms a partition of the domain X.

Let us prove this latter statement.

- Observe that each $f^{-1}(\{y\})$ is nonempty; indeed, if $y \in \mathrm{Im}(f)$, there must be some $x \in X$ such that $f(x) = y$ so that $x \in f^{-1}(\{y\})$.
- By part (a), the sets $f^{-1}(\{y\})$, $f^{-1}(\{y'\})$ are disjoint if $y \neq y'$.
- Finally, the same argument from part (b) proves that $\bigcup_{y \in \mathrm{Im}(f)} \{f^{-1}(\{y\}\}) = X$. Indeed, $\bigcup_{y \in \mathrm{Im}(f)} \{f^{-1}(\{y\}\}) \subseteq X$ is immediate and for the reverse inclusion, we note that for any $x \in X$, the value $y = f(x)$ is an element of $\mathrm{Im}(f)$ (by definition of the image), so $x \in f^{-1}(\{y\})$.

(d) We start by noting that $f : \mathbb{R} \to \mathbb{Z}$ is surjective; for any $n \in \mathbb{Z}$ we have $\lfloor n \rfloor = n$ (see Exercise 147). Therefore the collection of preimages $\{f^{-1}(\{n\}) : n \in \mathbb{Z}\}$ forms a partition of the domain $\mathbb{R}$.

The block $f^{-1}(\{n\})$ contains all the real numbers x such that $f(x) = \lfloor x \rfloor = n$. That is, all real numbers x such that the "greatest integer less than or equal to x is n". This clearly includes all real numbers in the interval $[n, n+1)$; i.e., $[n, n+1) \subseteq f^{-1}(\{n\})$.

We claim this inclusion is in fact an equality. Indeed, if $x < n$ then n does not satisfy the condition of being an "integer less than or equal to x" so $\lfloor x \rfloor \neq n$. Similarly, if $x \geq n+1$, then n does not satisfy the requirement of being the "*greatest* integer less than or equal to x" so that $\lfloor x \rfloor \neq n$. This proves that
$$f^{-1}(\{n\}) = [n, n+1).$$

Helpful Tip!

You may enjoy revisiting Exercise 8 from the Introduction to Functions handout, where you were asked to draw the graph of the floor functions. The equivalence classes are precisely the half-open intervals drawn on that graph! Indeed, if we project this interval onto the x-axis, we see that they perfectly partition the domain. This is precisely what the current exercise is claiming.

(Exercise on page 165.)