Section 10.4: Taylor Series and Taylor Polynomials

In this section, we will learn how to use polynomial functions to approximate other elementary functions.

Before we can find a polynomial function P to approximate another function f, we need to decide "where" we want the two functions to be similar. That is, our task is to find a polynomial whose graph resembles the graph of f near some point a.

One way to do this is require that

$$P(a) = f(a) \quad \text{and} \quad P'(a) = f'(a) .$$

With these two requirements, we can obtain a linear approximation of the function f. We say that the approximating polynomial is "*expanded about a* or *centered at a*."

The Picture:

Is there any way to improve our approximation? If we impose the additional requirement that $P''(a) = f''(a)$, will our approximation improve? How about higher derivatives?

Example: Find a 2nd degree polynomial $P_2(x)$ that approximates the function $f(x) = e^x$ near $x = 0$. Take a look at their graphs.

The polynomial approximation of $f(x) = e^x$ given above is expanded about $a = 0$. For expansions about an arbitrary value of *a*, it is convenient to write the polynomial in the form

$$P_n(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n .$$

In this form, repeated differentiation produces

$P_n'(x) =$

$P_n''(x) =$

$P_n'''(x) =$

$\vdots$

$P_n^{(n)}(x) =$

By letting $x = a$, we obtain

$$P_n(a) = c_0 \qquad P_n'(a) = c_1 \qquad P_n''(a) = 2c_2 \qquad \ldots \qquad P_n^{(n)}(a) = n!c_n$$

Since the value of *f* and its first *n* derivatives must agree with the value of P_n and its first *n* derivatives at $x = a$, it follows that

$$f(a) = c_0 \qquad f'(a) = c_1 \qquad \frac{f''(a)}{2!} = c_2 \qquad \ldots \qquad \frac{f^{(n)}(a)}{n!} = c_n$$

Using these coefficients, we obtain the following definition.

Definition of *n*th Taylor Polynomial and *n*th Maclaurin Polynomial: If *f* has *n* derivatives at *a*, then the polynomial

$$\boxed{P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n}$$

is called the *nth Taylor polynomial for f at c*. If $a = 0$, then we have the *nth Maclaurin polynomial for f*

$$\boxed{P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n}.$$

The Picture:

Example: Find the 5th degree Maclaurin polynomial for $f(x) = e^x$. Use this polynomial to approximate $e^{.01}$. How close are we?

Note: We can use $P_n(x)$ to approximate $f(x)$ "near" $x = a$. State 2 requirements for a good approximation.

1.

2.

Since $P_n(x)$ is only an approximation, there is an error term (also called the *remainder*):

$$\boxed{R_n(x) = f(x) - P_n(x)}.$$

Without justification, we state the following theorem.

Theorem 2 (Taylor's Formula): Suppose that the $(n+1)$th derivative of the function *f* exists on an interval *I* containing *a*. Then

$$\boxed{f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1}},$$

for some *z* between *a* and *x* (where *x* is also in the interval *I*).

That is, we have

$$\boxed{R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1}},$$

for some *z* between *a* and *x*.

Example: Compute Taylor's Formula for $f(x) = \ln x$ at $a = 1$ with $n = 4$.

Note: What happens if $n = 0$?

Now, suppose that the function *f* has derivatives of all orders. Also, suppose that $R_n(x)$ gets smaller as *n* gets larger (in fact, we want $\lim_{n\to\infty} R_n(x) = 0$). Then we can define the *Taylor Series of the function f at* $x = a$:

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n}.$$

If we let $a=0$ above, then we get the *Maclaurin series of the function f at* $x=a$.

Some Maclaurin Series Worth Mentioning:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

Note:

1. Cosine is an ______ function.
2. Sine is an ______ function.

Example:

(a) Derive the Maclaurin Series for $f(x)=\cos x$.

(b) Using one of the known Maclaurin formulas, find the Maclaurin series of $f(x)=e^{2x}$.