Section 9.5: Integral Computations with Parametric Curves

Recall: In Calculus I, the following formulas were introduced:

- Area under the curve:

$$A = \int_a^b y\,dx$$

- Volume of revolution around x-axis:

$$V = \pi \int_a^b y^2 dx$$

- Volume of revolution around y-axis:

$$V = 2\pi \int_a^b xy\, dx$$

- Arc length:

$$s = \int_a^b \sqrt{1+\left[dy/dx\right]^2}\,dx$$

- Area of the surface of revolution around x-axis:

$$S = 2\pi \int_a^b y\sqrt{1+\left[dy/dx\right]^2}\,dx$$

- Area of the surface of revolution around y-axis:

$$S = 2\pi \int_a^b x\sqrt{1+\left[dy/dx\right]^2}\,dx$$

If $x = f(t)$ and $y = g(t)$, then we get

$$dx = f'(t)dt$$
$$dy = g'(t)dt$$
$$\sqrt{1+\left[dy/dx\right]^2}\,dx = \sqrt{1+\left[g'(t)/f'(t)\right]^2}\,f'(t)dt = \sqrt{f'(t)^2 + g'(t)^2}\,dt$$

By making the appropriate substitutions, we can find area, volume of revolution, arc length, and surface area of smooth parametric curves.

Important:

1. To evaluate integrals involving area or volume of revolution (which involve dx), we integrate either from $t = \alpha$ to $t = \beta$ <u>or</u> from $t = \beta$ to $t = \alpha$; the proper choice of limits on t being the one that corresponds to traversing the curve in the positive x-direction from left to right.

2. On all of the integrals, we need to make sure that the curve is traced out only once of the interval of integration.

Example:

(a) Find the area of the region that lies between the given parametric curves and the x-axis.

$$x = e^{3t},\ y = e^{-t};\ 0 \le t \le \ln 2$$

(b) Find the volume obtained by revolving around the x-axis the region described in part (a).

(c) Find the arc length of the given curve.

$$x = r\cos t\,,\ \ y = r\sin t\ ;\ 0 \le t \le 2\pi$$

(d) Find the area of the surface of revolution generated by revolving the region determined by part (c) about the x-axis. Consider just $0 \le t \le \pi$.