Math 1300: Calculus I, Fall 2006
Review for Midterm Exam 2
Answers

I. True or False?

(a) False. (Not defined at 0)

(b) False. (The limit could exist, but not be equal to $f(c)$. Give a counterexample.)

(c) True. (This is the contrapositive of Theorem 3.2.3)

(d) False. (To apply the IVT we also need that f is continuous on $[0, 2]$. Give a counterexample.)

(e) False. (To apply the Squeezing Theorem we also need that $\lim_{x\to c} h(x) = \lim_{x\to c} g(x)$. Give a counterexample.)

(f) False. (If $f(x)$ is differentiable at $x = 0$, then $\lim_{h\to 0} \dfrac{f(h) - f(0)}{h}$ exists. $f(0)$ may not be equal to 0. Give a counterexample.)

(g) True. (Use the quotient rule to show this.)

(h) False. (Use the Chain Rule to find $\dfrac{d}{dx}[f(cx)]$. Give a counterexample.)

(i) False. (Give a counterexample.)

II. Recommended Problems from the Textbook

For the QuickCheck exercises, see the answers at the end of the homework section. For the odd-numbered problems, see the back of the textbook. Here are the answers to the even-numbered problems.

Section 2.5:

28. (a) Setting $f(c) = \lim_{x\to c} f(x)$ would remove the discontinuity.

    (b) $f(x)$ has a removable discontinuity at $x = 1$, because $\lim_{x\to 1} f(x)$ exits (and is equal to 2), but f is not defined at 1. $g(x)$ has a removable discontinuity at $x = 1$ because $\lim_{x\to 1} g(x)$ exists (check left- and right-hand limits), but $\lim_{x\to 1} g(x) \neq g(1)$

    (c) To remove the discontinuities, set $f(1) = 2$ and $g(1) = 1$.

38. f and g are continuous on [a, b], so $f - g$ is continuous on this interval as well. Since $f(a) > g(a)$, we have that $f(a) - g(a) > 0$. Similarly, $f(b) < g(b)$ gives $f(b) - g(b) < 0$. So by the Intermediate Value Theorem, we know that there exists at least one number c in (a, b) such that $f(c) - g(c) = 0$. But this implies that $f(c) = g(c)$, as desired.

Section 2.6:

14. f is continuous on $(-3, 0) \cup (0, \infty)$.

46. $k = 3$

Section 3.3:

22. $\left.\dfrac{dy}{dx}\right|_{x=1} = 0$

Section 3.5:

16. $f'(x) = 0$
22. $\dfrac{dy}{dx} = 2x\cos(x) - x^2\sin(x) + 4\cos(x)$
36. A pattern emerges after the first four or five derivatives:

$$\begin{aligned}
f(x) &= x\sin(x)\\
\frac{df}{dx} &= x\cos(x) + \sin(x)\\
\frac{d^2f}{dx^2} &= -x\sin(x) + 2\cos(x)\\
\frac{d^3f}{dx^3} &= -x\cos(x) - 3\sin(x)\\
\frac{d^4f}{dx^4} &= x\sin(x) - 4\cos(x)\\
\frac{d^5f}{dx^5} &= x\cos(x) + 5\sin(x)\\
&\vdots\\
\frac{d^{17}f}{dx^{17}} &= x\cos(x) + 17\sin(x)
\end{aligned}$$

44. Note that this limit is equal to the limit definition of the derivative of $\tan(y)$. So, the answer is $\sec^2(y)$.

Section 3.6:

14. $y' = \frac{1}{2}(\sqrt{x})^{\frac{-1}{2}}(\frac{1}{2}x^{\frac{-1}{2}})$
32. $y' = \cos(\tan(3x))\sec^2(3x)(3)$
54. $y' = \tan\left(\dfrac{1}{x}\right) + x\sec^2\left(\dfrac{1}{x}\right)\left(-\dfrac{1}{x^2}\right)$

Section 4.1:

24. $\dfrac{d^2y}{dx^2} = \dfrac{2xy + 2y^2}{(x+2y)^3} = \dfrac{4}{(x+2y)^3}$ since $xy + y^2 = 2$.

Section 4.2:

38. (b) Using the change of base formula, we can write $\log_{\ln x} e = \dfrac{\ln e}{\ln(\ln x)}$. Differentiating, we obtain $\dfrac{-1}{x\ln x(\ln(\ln x))^2}$.

44. Let $a > 0$ be an arbitrary x-value. Then $(a, \ln a)$ is a point on the graph of $y = \ln x$. The slope of the tangent line to the graph of $y = \ln x$ at $x = a$ is

$$\left.\frac{dy}{dx}\right|_{x=a} = \frac{1}{a}.$$

This implies that the equation of the line tangent to $y = \ln x$ at $x = a$ is

$$y - \ln a = \frac{1}{a}(x - a),$$

which becomes

$$y = \frac{x}{a} + \ln a - 1.$$

Note that the y-intercept of this line is $\ln a - 1$. So, the y-intercept is 1 less than the y-value of the point of tangency.

III. Additional Problems

1. f is not continuous at $x = 0, 1$.

2. This problem is "broken." There is no such k. The trouble is that there will always be a discontinuity at $x = 1$ for any value of k. If $k = 4$, the function will be continuous at $x = 5$. But this doesn't fix the discontinuity at $x = 1$.

3. No. $f(x)$ is a rational function so it will be continuous on $[0, 1]$ if and only if the denominator is never zero on that interval. Let $g(x) = x^5 + \pi x - e$. Since $g(x)$ is a polynomial it is continuous on $[0, 1]$. $g(0) < 0$ and $g(1) > 0$, so the Intermediate Value Theorem guarantees the existence of at least one point c in $(0, 1)$ such that $g(c) = 0$. Thus $f(x)$ is not continuous on $[0, 1]$.

4. $\lim_{x \to 1} f(x) = 3$

5. First, note that $-1 \le \cos(2/x) \le 1$ (for $x \ne 0$). This implies that $-x^4 \le x^4 \cos(2/x) \le x^4$ (for $x \ne 0$). Now, note that

$$\lim_{x \to 0}(-x^4) = 0 = \lim_{x \to 0}(x^4).$$

By the Squeezing Theorem,

$$\lim_{x \to 0}(x^4 \cos(2/x)) = 0.$$

6. (a) $v_{\text{ave}} = -122$ ft/sec
   (b) $v_{\text{inst}} = h'(2) = -90$ ft/sec
   (c) The distance between our representative and the ground is decreasing, i.e., he/she is gonna hit the ground.

7. (a) Using the definition of the derivative, we get

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{((x+h)^2-(x+h))-(x^2-x)}{h} \\
&= \lim_{h\to 0} \frac{x^2+2xh+h^2-x-h-x^2+x}{h} \\
&= \lim_{h\to 0} \frac{2xh+h^2-h}{h} \\
&= \lim_{h\to 0}(2x+h-1) \\
&= 2x-1
\end{aligned}$$

(b) $y-2=3(x-2)$

8. $\dfrac{dy}{dx} = 2x\sin\left(\dfrac{\pi}{4}x\right) + \dfrac{\pi}{4}x^2\cos\left(\dfrac{\pi}{4}x\right)$

9. False, you must use the product rule to find this derivative. The correct derivative is:

$$\frac{d}{dx}[(2x^3+5x^2-7)(3\cos(x)+13x)] = (2x^3+5x^2-7)(-3\sin(x)+13)+(3\cos(x)+13x)(6x^2+10x)$$

10. $\dfrac{d}{dx}[\csc(\csc(\sin(x)))] = (\cot(\csc(\sin(x))))(\csc(\csc(\sin(x))))(\cot(\sin(x)))(\csc(\sin(x)))(\cos(x))$

11. $\sec(x)\tan^2(x)+\sec^3(x)-2\cos(x)\sin(x)$

12. $y-2=2(x-\pi/4)$

13. Using the limit definition of the derivative, we get

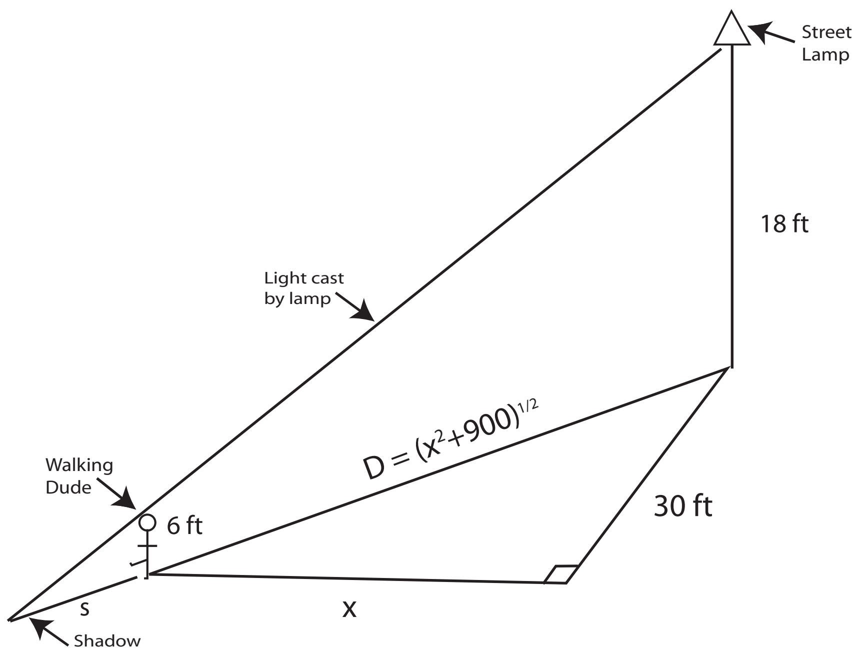
$$\begin{aligned}
\frac{d}{dx}[\cos(x)] &= \lim_{h\to 0}\frac{\cos(x+h)-\cos(x)}{h} \\
&= \lim_{h\to 0}\frac{\cos(x)\cos(h)-\sin(h)\sin(x)-\cos(x)}{h} \\
&= \lim_{h\to 0}\left(\cos(x)\frac{\cos(h)-1}{h}-\sin(x)\frac{\sin(h)}{h}\right) \\
&= \left(\lim_{h\to 0}\cos(x)\right)\left(-\lim_{h\to 0}\frac{1-\cos(h)}{h}\right)-\left(\lim_{h\to 0}\sin(x)\right)\left(\lim_{h\to 0}\frac{\sin(h)}{h}\right) \\
&= (\cos(x))(0)-(\sin(x))(1) \\
&= -\sin(x)
\end{aligned}$$

14. $(f\circ g\circ h)'(1)=48$

15. $V=\frac{4}{3}\pi r^3$. So $\dfrac{dV}{dt}=4\pi r^2\dfrac{dr}{dt}$. We know that $\dfrac{dV}{dt}=-3$ in^3/sec and are interested in the point when $r=2$. Plugging these values in gives $\dfrac{dr}{dt}=\dfrac{-3}{16\pi}$ in/sec. If D is the diameter of the balloon, we have $D=2r$, so $\dfrac{dD}{dt}=2\dfrac{dr}{dt}$, giving us that $\dfrac{dD}{dt}=\dfrac{-3}{8\pi}$ in/sec.

16. Let θ represent the acute angle the ladder makes with the ground, and let h be the height of the top of the ladder. We have $\sin(\theta) = \frac{h}{10}$, or $10\sin(\theta) = h$.
Differentiating gives $10\cos(\theta)\frac{d\theta}{dt} = \frac{dh}{dt}$. We know that $\frac{dh}{dt} = -2$ ft/sec and are interested in the point when $h = 5$. Plugging in these values gives $\frac{d\theta}{dt} = \frac{-2}{5\sqrt{3}}$ rad/sec.

17. There are two triangles in this problem (see figure below).



The first is the triangle in the plane of the road. Let x be the distance from the man to the point directly across from the lamp. Then we have a right triangle with legs x and 30 because the road is 30 ft wide. The straight-line distance along the road from the lamp to the man is given by the Pythagorean Theorem, so we have $D = \sqrt{x^2 + 900}$. Now, there's also the triangle we have whose hypotenuse is from the light to the end of the shadow. The total distance along the ground is given by the distance from the light to the man plus the length of his shadow. Letting s be his shadow length, we have that the leg is $s + \sqrt{x^2 + 900}$. Notice that the man is inside this triangle and the man and his shadow form a similar triangle to the light plus that whole length we just found. Using the fact that corresponding parts of congruent triangles are proportional we get

$$\frac{6}{s} = \frac{18}{s + \sqrt{x^2 + 900}}.$$

We can simplify this to get the following relationship between x and s: $2s = \sqrt{x^2 + 900}$. Now differentiating both sides gives us

$$2\frac{ds}{dt} = \frac{2x}{2\sqrt{x^2 + 900}}\frac{dx}{dt}.$$

We know that $\frac{dx}{dt} = 5$ ft/sec. Also, we are interested in the point where $x = 40$ ft. We can plug these values in to get $\frac{ds}{dt} = 2$ft/sec.

18. (a) Let $y = \sin^{-1}(x)$ and apply sine to both sides. You get

$$\sin(y) = \sin(\sin^{-1}(x)) = x.$$

Now we can use implicit differentiation on $\sin(y) = x$. You get $\cos(y)\frac{dy}{dx} = 1$ and solving for $\frac{dy}{dx}$ you get:

$$\frac{dy}{dx} = \frac{1}{\cos(y)}.$$

You are NOT DONE yet though! Now we must write $\cos(y)$ in terms of x. We know that $\cos^2(y) + \sin^2(y) = 1$ so solving for $\cos(y)$ we get

$$\cos(y) = \sqrt{1 - \sin^2(y)}.$$

Now, we know that $\sin(y) = x$ so we can replace the $\sin^2(y)$ in the square root by x^2. Hence we have that

$$\frac{dy}{dx}[\sin^{-1}(x)] = \frac{1}{\sqrt{1-x^2}}.$$

(b) We know that the equation of the tangent line to $f(x) = \sin^{-1}(x)$ at $c = \frac{1}{2}$ has the form

$$y - f(c) = f'(c)(x - c).$$

To find $f(c)$ we just plug in $c = \frac{1}{2}$ into f and we get $f(\frac{1}{2}) = \sin^{-1}(\frac{1}{2}) = \frac{\pi}{6}$.
To find $f'(c)$ we just plug in $c = \frac{1}{2}$ into what we got in part (a) and we get:

$$f'(1/2) = \frac{1}{\sqrt{1-(\frac{1}{2})^2}} = \frac{1}{\sqrt{1-\frac{1}{4}}} = \frac{1}{\sqrt{\frac{3}{4}}} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}}.$$

So the equation of the tangent line is

$$y - \frac{\pi}{6} = \frac{2}{\sqrt{3}}(x - \frac{1}{2}).$$

or

$$y = \frac{2}{\sqrt{3}}x - (\frac{1}{\sqrt{3}} - \frac{\pi}{6})$$

19. (a) $f'(x) = e^{e^{e^{e^x}}} e^{e^{e^x}} e^{e^x} e^x.$

(b) $f'(x) = \frac{\log(x)^{\log(x)}}{x}\left(\frac{1}{\ln 10} + \log(\log(x))\right)$

20. (a) $f'(x) = \frac{(x^2+3)(1) - (x)(2x)}{(x^2+3)^2} = \frac{x^2+3-2x^2}{(x^2+3)^2} = \frac{-x^2+3}{(x^2+3)^2}$

(b) $f'(x) = (1)(x^2+3)^{-1} + (x)(-(x^2+3)^{-2}(2x)) = \frac{1}{x^2+3} + \frac{x(2x)}{-(x^2+3)^2}$
$= \frac{x^2+3}{(x^2+3)^2} - \frac{2x^2}{(x^2+3)^2} = \frac{-x^2+3}{(x^2+3)^2}$

(c) Let $y = \dfrac{x}{x^2+3}$. Then $\ln(y) = \ln(x) - \ln(x^2+3)$. Differentiating both sides gives

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x} - \frac{1}{x^2+3}(2x).$$

Simplifying and multiplying both sides by y gives us

$$\frac{dy}{dx} = y\left(\frac{-x^2+3}{x(x^2+3)}\right)$$

Finally, plugging $\dfrac{x}{x^2+3}$ back in for y and simplifying we get

$$\frac{dy}{dx} = \frac{x}{x^2+3}\left(\frac{-x^2+3}{x(x^2+3)}\right) = \frac{-x^2+3}{(x^2+3)^2}$$

21. (a) $\dfrac{dy}{dx} = \left(\dfrac{1}{x^2}\right)2x = \dfrac{2}{x}$

(b) $\dfrac{dy}{dx} = \ln 2$

(c) $\dfrac{dy}{dx} = ex^{e-1}$

(d) Let's assume that $x > 0$, otherwise there are problems. First, take the natural log of both sides: $\ln y = \ln x^x = x \ln x$.

Then taking the derivative of both sides gives peace of mind and $\dfrac{1}{y}\dfrac{dy}{dx} = \ln x + 1$.

Sub y back in to get $\dfrac{dy}{dx} = x^x(\ln x + 1)$.

(e) $\dfrac{dy}{dx} = 2^x \ln(2)$

(f) $\dfrac{dy}{dx} = -\tan(x)$

(g) $\dfrac{dy}{dx} = 3e^{3x+5}$

(h) $\dfrac{dy}{dx} = (11x^2+9x-7)^{\ln(x)}\left(\dfrac{\ln(11x^2+9x-7)}{x} + \dfrac{\ln(x)(22x+9)}{11x^2+9x-7}\right)$

(i) $\dfrac{dy}{dx} = \dfrac{3(\ln x)^2}{x}\left(x^{\ln x}\right)^{\ln x}$

22. (a) $f'(x) = \dfrac{(3-x)^{1/3}x^2}{(1-x)(3+x)^{2/3}}\left(\dfrac{-1}{3(3-x)} + \dfrac{2}{x} + \dfrac{1}{1-x} - \dfrac{2}{3(3+x)}\right)$.

(b) $f'(x) = 2e^{x^2}x(x+1) + e^{x^2} + 1$.