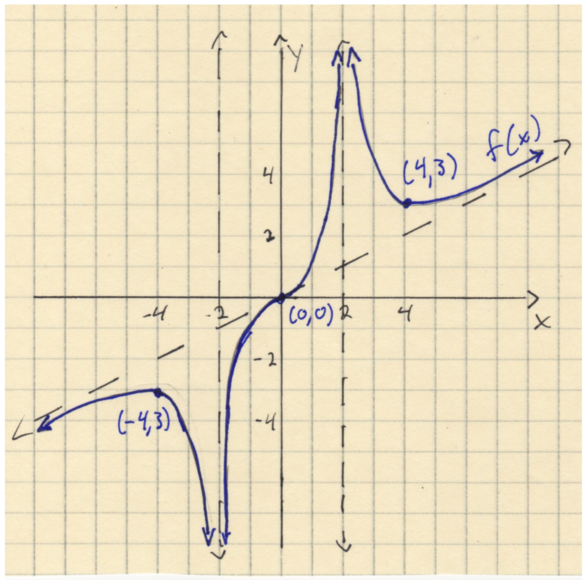
Math 1300: Calculus I, Fall 2006
Answer Sheet for Review 3

1. Your graph should look something like this:



2. No, $\lim_{x\to 0} \frac{f'(x)}{g'(x)}$ does not exist.

3. 11.5 and 11.5

4. Answer: $\dfrac{1}{6}$.

5. (a) $\displaystyle\sum_{k=0}^{n} \left(\frac{1}{2^k}\right)^3 = \sum_{k=0}^{n} \frac{1}{8^k}$

   (b) $\displaystyle\lim_{n\to\infty} \frac{8\,(8^n - 1/8)}{7\,(8^n)} = \frac{8}{7}$ cm^3

6. Solution:

   (1) Increasing: $(-\infty, -2] \cup [0, \infty)$
   Decreasing: $[-2, 0]$

   (2) Relative Maximum: $(-2, 4e^{-2})$
   Relative Minimum: $(0, 0)$

   (3) Concave Up: $(-\infty, -2-\sqrt{2}) \cup (-2+\sqrt{2}, +\infty)$
   Concave Down: $(-2-\sqrt{2}, -2+\sqrt{2})$
   Inflection Points: $x = -2 \pm \sqrt{2}$

7. $f(x)$ attains its minimum of 1 at $x = 0$. By the definition of an absolute minimum, $f(x) \geq 1 > 0$ for all x. The desired inequality follows directly from this.

8. To prove the first statement in the Hint, suppose that the graph of f has a point $P = (b, f(b))$ with b in I such that P lies below or on the tangent line ℓ and $a < b$. Then

$$\frac{f(b) - f(a)}{b - a} \leq (\text{the slope of } \ell) = f'(a).$$

Since f is differentiable on the open interval I and a, b are in I, we get that f is continuous on $[a, b]$ and differentiable on (a, b). Hence by the Mean-Value Theorem

$$\frac{f(b) - f(a)}{b - a} = f'(c) \qquad \text{for some } c \text{ in } (a, b).$$

Thus $f'(a) \geq f'(c)$ holds for this c in (a, b).
The proof of the second statement in the Hint is similar.
These two statements show that if some point of the graph of f on the interval I is below or on the line ℓ, then f' is not increasing on I, that is, f is not concave up on I. Hence, if f is concave up on I, then every point of the graph of f on I is above the line ℓ.

9. Note that f is not continuous on $[-1, 2]$. Now, $f'(x) = 1 - x^{-2}$ and $\frac{f(2)-f(-1)}{2-(-1)} = \frac{3}{2}$, but there is no solution to $1 - x^{-2} = \frac{3}{2}$ in the real numbers.

10. False, as $f(x) = e^x$ is a counterexample.

11. $\int_1^{11} f(x)dx = \int_0^{11} f(x)dx - \int_0^1 f(x)dx = 29 - (-7) = 36$.

12. $\int_0^3 e^{1+x} dx$.

13. Critical points at $x = 0$ and $x = 1$, absolute maximum at $x = 0$, absolute minimum at $x = -1$.

14. Answer: 1.

15. $f(x)$ is increasing on $(-\infty, +\infty)$ since $f'(x)$ is always positive.

16. $f(x)$ has an inflection point at $x = a$ if n is odd and ≥ 3.

17. SYMMETRIES: none

x-INTERCEPT: $x = 0$

y-INTERCEPT: $y = 0$

VERTICAL ASYMPTOTE: The line $x = 1$

Horizontal asymptote: Since $\lim_{x\to+\infty} f(x) = 0$ and $\lim_{x\to-\infty} f(x) = 0$, the line $y = 0$ is a horizontal asymptote.

Derivatives: $f'(x) = \dfrac{1+2x}{3x^{2/3}(1-x)^2}$ and $f''(x) = \dfrac{2(5x^2+5x-1)}{9x^{5/3}(1-x)^3}$

Critical points: $x = -1/2$ and $x = 0$

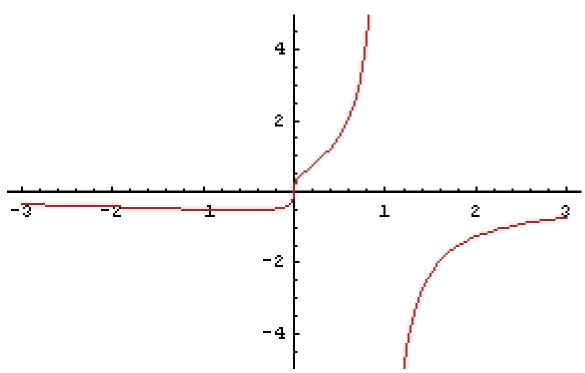
Increasing/Decreasing behavior: $f(x)$ is decreasing on the interval $(-\infty, -1/2]$, and increasing on the intervals $[-1/2, 1)$ and $(1, +\infty)$.

Relative extrema: $f(x)$ has a relative minimum at $x = -1/2$

Concavity: The numbers for which $f''(x) = 0$ are $-\frac{1}{2} + \frac{3\sqrt{5}}{10}$ and $-\frac{1}{2} - \frac{3\sqrt{5}}{10}$; f'' is undefined for $x = 0$ and discontinuous at $x = 1$. Checking the sign of f'' on the intervals determined by these numbers, we get that $f(x)$ is concave down on $\left(-\infty, -\frac{1}{2} - \frac{3\sqrt{5}}{10}\right)$, concave up on $\left(-\frac{1}{2} - \frac{3\sqrt{5}}{10}, 0\right)$, concave down on $\left(0, -\frac{1}{2} + \frac{3\sqrt{5}}{10}\right)$, concave up on $\left(-\frac{1}{2} + \frac{3\sqrt{5}}{10}, 1\right)$, and concave down on $(1, \infty)$.

Inflection points: $x = -\frac{1}{2} - \frac{3\sqrt{5}}{10}$, $x = 0$, and $x = -\frac{1}{2} + \frac{3\sqrt{5}}{10}$

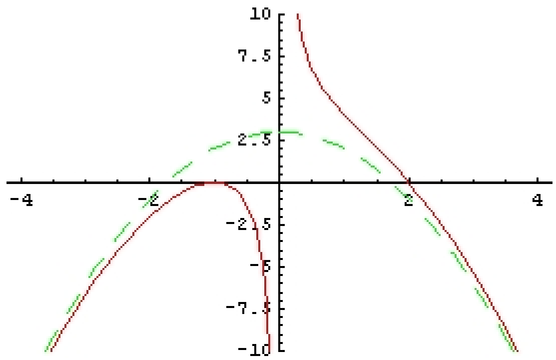
Your graph should look like this:



ANSWER TO #18 IN SECTION 5.3:

$$\frac{2+3x-x^3}{x} - (3-x^2) = \frac{2}{x} \to 0 \text{ as } x \to \pm\infty.$$

The graph should look like



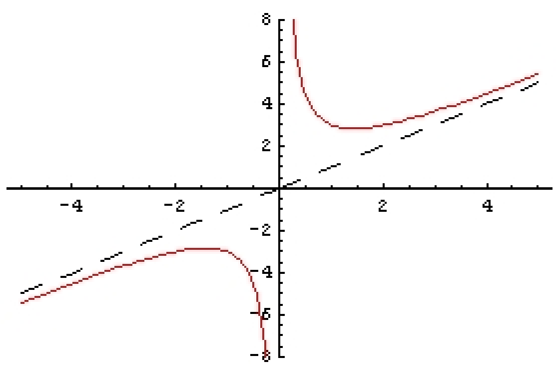
where the dotted green curve is the curvilinear asymptote.

18. 30 km from B.

19. Here are the answers:

(1) The x-intercepts are $(\sqrt{2}, 0)$ and $(-\sqrt{2}, 0)$. There are no y-intercepts.
(2) There are no critical points since $f'(x)$ is never zero and $x = 0$ (where there derivative is undefined) is not in the domain of f.
(3) f is increasing on $(-\infty, 0)$ and $(0, \infty)$. f is concave up on $(-\infty, 0)$ and concave down on $(0, \infty)$.
(4) There is a vertical asymptote $x = 0$ and an oblique asymptote $y = x$.

The graph should look like:



Answers for True or False:

(a) False.
For example, $\lim_{x\to\infty} \frac{x + \sin(x)}{x}$ is an indeterminate form of type ∞/∞ such that $\lim_{x\to\infty} \frac{x + \sin(x)}{x} = 1$ but $\lim_{x\to\infty} \frac{\frac{d}{dx}(x + \sin(x))}{\frac{d}{dx}x} = \lim_{x\to c\infty} \frac{1 + \cos(x)}{1}$ does not exist.
(b) False.
For example, $f(x) = x^3$ is increasing and differentiable on $(-\infty, \infty)$, but $f'(0) = 0$.

(c) True.
If f is not increasing on I, then there exist $a < b$ in I such that $f(a) \geq f(b)$. Since f is differentiable on I, it follows that f is continuous on $[a, b]$ and differentiable on (a, b). As f is not increasing on $[a, b]$, we get from Theorem 5.1.2 that $f'(x) \not> 0$ for some x in I.

(d) False.
The function $f(x) = x^3$ does not have a relative extremum, but $f'(0) = 0$.

(e) False.
The function $f(x) = \sqrt[3]{x}$ has an inflection point at $x = 0$, but $f''(0)$ is not defined.

(f) True.
If $f''(x)$ is continuous on I and $f''(x) \neq 0$ for all x in I, then by the Intermediate-Value Theorem $f''(x)$ does not change sign on I. Hence either $f''(x) > 0$ for all x in I, so f is concave up on I, or $f''(x) < 0$ for all x in I, so f is concave down on I.

(g) True.
If $f(x) = \dfrac{P(x)}{Q(x)}$ is a rational function, then long division yields polynomials $q(x)$ and $r(x)$ such that $P(x) = q(x)Q(x) + r(x)$ and the degree of $r(x)$ is less than the degree of $Q(x)$. Hence

$$f(x) = \frac{q(x)Q(x) + r(x)}{Q(x)} = q(x) + \frac{r(x)}{Q(x)}$$

where

$$\lim_{x\to\infty} \frac{r(x)}{Q(x)} = 0 = \lim_{x\to-\infty} \frac{r(x)}{Q(x)}.$$

Thus $y = q(x)$ is an asymptote of f.

(h) False.
$f(x) = \sqrt[3]{x}$ has a vertical tangent line at $x = 0$, but it has no vertical asymptote since it is continuous everywhere.

(i) True.
Define a function h by $h(x) = g(x) - 2f(x)$ for all x in $(-\infty, \infty)$. The assumptions imply that $h(0) = 0$ and $h'(x) = g'(x) - 2f'(x) = 0$ for all x in $(-\infty, \infty)$. Therefore by the Constant Difference Theorem $h(x) = 0$ for all x in $(-\infty, \infty)$. Hence $g(x) = 2f(x)$ for all x in $(-\infty, \infty)$.

(j) False.
Let

$$f(x) = \begin{cases} 1/x & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Then f is differentiable on $(0, 1)$, $f'(x) < 0$ for all x in $(0, 1)$, and $\dfrac{f(1) - f(0)}{1 - 0} = 1 > 0$. Therefore there is no c in $(0, 1)$ such that $\dfrac{f(1) - f(0)}{1 - 0} = f'(c)$.