Math 1300: Calculus I, Fall 2006
Review for Midterm Exam 3

Exam 3 covers sections 4.4, 5.1–5.5, 5.7, 6.1, 6.4 and 6.5 out of our textbook. Do the following problems (including the problems marked with a • from our textbook) as part of your review. Be ready to state theorems and definitions on the exam.

1. Suppose $f(x) = P(x)/Q(x)$ is a rational function where $P(x)$ and $Q(x)$ share no common factors. Suppose further that $f(x)$ satisfies the following properties:

(1) $f'(x) > 0$ on $(-2, 2)$
(2) $f(x)$ has a local maximum at the point $(-4, -3)$ and a local minimum at the point $(4, 3)$
(3) $f''(x) < 0$ on $(-\infty, -2) \cup (-2, 0)$
(4) $f''(x) > 0$ on $(0, 2) \cup (2, \infty)$
(5) $f(x)$ is symmetric about the origin
(6) $f(x)$ has an oblique asymptote at $y = x/2$ (or $y = \frac{1}{2}x$)
(7) $P(x)$ has a root of multiplicity 3 at $x = 0$
(8) $Q(x)$ has roots at $x = 2$ and $x = -2$.

Sketch a graph of $f(x)$, labelling any important points.

• Now do #1 in the Quick Check and #23 in the Exercises from section 5.3

2. Let $f(x) = |x|$ and $g(x) = \sin x$. Can L'Hôpital's Rule be used to evaluate $\lim_{x\to 0} \frac{f(x)}{g(x)}$? If so, what is the limit? If not, justify your answer.

• Now do #35 and #39 in section 4.4.

3. Find two numbers whose sum is 23 and whose product is a maximum.

• Now do #5 and #9 in section 5.5.

4. Evaluate $\lim_{N\to\infty} \sum_{k=1}^{N} \frac{(k+2)(k+1)}{N(N+1)(2N+1)}$.

• Now do #23 and #47 from section 6.4.

5. In this problem, we will find the volume of a certain infinite stack of cubes. Start with a cube that has sides of length 1 cm. On top of the first cube stack a second cube that has sides of length 1/2 cm. On top of the second cube stack a third cube that has sides of length 1/4 cm. Continue in this way, so that the length of the sides of each new cube is 1/2 the length of the sides of the cube just below.

(a) Using sigma notation, write a formula for the volume of the first n cubes.

(b) Now, using the fact below, find the volume of the infinite stack of cubes described above. Fact: $\sum_{k=0}^{n} \frac{1}{r^k} = \frac{r}{r-1}\left(1 - \frac{1}{r^{n+1}}\right)$, where r is a positive integer strictly greater than 1. This is a special case of a sum called *geometric sum.*

- Now do #47 and #59(a) in section 6.4.

6. Let $f(x) = x^2 e^x$.

(1) Find the intervals on which f is increasing or decreasing.
(2) Find the critical points of f and determine for each point whether it is a relative maximum, relative minimum, or neither.
(3) Find the intervals of concavity and the inflection points of f.

- Now do #1 in section 5.2.

7. Verify that $e^{x^2} > x^2$ by showing that the absolute minimum of $f(x) = e^{x^2} - x^2$ is strictly greater than zero. Why does this prove the desired inequality for all x?

- Now do #11 and #19 in section 5.4.

8. Let f be differentiable on an open interval I, and let ℓ denote the tangent line to the graph of f at $x = a$ for some a in I. Use the Mean-Value Theorem to show that if f is concave up on I, then on the interval I the graph of f is above the line ℓ.

Hint: Suppose that the graph of f has a point $P = (b, f(b))$ with b in I such that P lies below or on the tangent line ℓ. Argue that

(1) if $a < b$, then there exists c in (a, b) such that $f'(a) \geq f'(c)$;
(2) if $b < a$, then there exists c in (b, a) such that $f'(c) \geq f'(a)$.

- Now do #21 and #23(a) in section 5.7.

9. Let $f(x) = x + \frac{1}{x}$ on $[-1, 2]$. Show that there is no point c such that $f'(c) = \frac{f(b)-f(a)}{b-a}$ on the given interval. Which hypothesis of the mean value theorem fails to hold?

- Now do #3 and #11 in section 5.7.

10. True or False? Justify your answer: If $f''(x)$ is defined and positive for all x, then $f(x)$ has exactly one relative minimum.

- Now do #17 and #23 in section 5.2.

11. Find $\int_1^{11} f(x)dx$ if $\int_0^1 f(x)dx = -7$ and $\int_0^{11} f(x)dx = 29$.

• Now do #11 in section 6.5.

12. Write

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{3}{n} e^{1+3k/n}$$

as a definite integral. Do not evaluate the integral.

• Now do #21 and #25 in section 6.5.

13. List all critical points and find the absolute maximum and minimum values of $f(x) = 2x - 3x^{2/3}$ on the interval $[-1, 3]$.

• Now do #13 in section 5.4.

14. Evaluate $\lim\limits_{x\to 0^+} x^x$.

• Now do #11 and #21 in section 4.4.

15. Find the intervals on which $f(x) = x^{17} + x^9 + x + 1$ is increasing and decreasing.

• Now do #5 and #7 in section 5.1.

16. Suppose that a is constant and n is a positive integer. What can you say about the existence of inflection points of the function $f(x) = (x - a)^n$? Justify your answer.

• Now do #23 in section 5.1.

17. Sketch the graph of $f(x) = \dfrac{x^{\frac{1}{3}}}{1-x}$.

• Now do #18 in section 5.3. The answer will be provided on the answer sheet since it is an even-numbered problem.

18. A dune buggy is in the desert at a point A located 40 km from a point B. There is a 50 km road connecting B to a point D that is perpendicular to the line connecting A to B. The dune buggy is trying to get to point D as quickly as possible. It can cut across the desert and get on the road to D at any point between B and D. If the dune buggy can travel at 45 km/h on the desert and 75 km/h on the road, how far from B should the dune buggy get on the road to D to minimize time?

• Now do #21 and #43(a) in section 5.5.

19. Here is a function and its derivatives.

$$f(x) = \frac{x^2-2}{x} \qquad f'(x) = \frac{x^2+2}{x^2} \qquad f''(x) = \frac{-4}{x^3}.$$

Do the following:

(1) Find all intercepts.
(2) Find all critical points and list if they are relative maxima, relative minima, cusps, vertical tangent lines, or none of these.
(3) List the intervals where $f(x)$ is increasing, decreasing, concave up, and concave down
(4) List, and give equations for, any asymptotes. (Vertical, Horizontal, Oblique or curvilinear)
(5) Graph $f(x)$.

• Also do #15, #17 and #19 in section 6.1.

True or False? Justify your answer.

(a) If f and g are differentiable on an open interval containing c, except possibly at $x = c$, and $\lim_{x\to c} \frac{f(x)}{g(x)}$ is an indeterminate form of type ∞/∞, then $\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$.
(b) If f is increasing and differentiable on (a, b), then $f'(x) > 0$ for all x in (a, b).
(c) If f is differentiable, but not increasing on an open interval I, then $f'(x) \leq 0$ for some x in I.
(d) If $f'(c) = 0$, then f has a relative extremum at $x = c$.
(e) If f has an inflection point at $x = c$, then $f''(c) = 0$.
(f) If $f''(x)$ is continuous on an open interval I and $f''(c) \neq 0$ for all $x \in I$, then f is either concave up on I or f is concave down on I.
(g) Every rational function has an asymptote that is a polynomial function.
(h) If a function has a vertical tangent line, then it has a vertical asymptote.
(i) If f and g are differentiable everywhere, both have y-intercept 0, and $g'(x) = 2f'(x)$ for all x in $(-\infty, \infty)$, then $g(x) = 2f(x)$ for all x in $(-\infty, \infty)$.
(j) If f is differentiable on the open interval (a, b), then there exists c in (a, b) such that $\frac{f(b) - f(a)}{b - a} = f'(c)$.