MATH 1300: CALCULUS 1

September 20, 2006

MIDTERM 1

I have neither given nor received aid on this exam.

Name:____

- ◯ **001** L. Mayhew .... (8am)
- ◯ **002** J. Kish .... (8am)
- ◯ **003** P. Newberry .... (8am)
- ◯ **004** A. Spina .... (8am)
- ◯ **005** M. Hedges .... (9am)
- ◯ **006** M. Stackpole .... (9am)
- ◯ **007** E. Mankin .... (9am)
- ◯ **008** N. Flores .... (9am)
- ◯ **009** A. Szendrei .... (9am)
- ◯ **010** A. Szendrei .... (10am)
- ◯ **012** J. Fuhrmann .... (11am)
- ◯ **013** B. Wang .... (11am)
- ◯ **014** D. Ernst .... (12pm)
- ◯ **015** C. Moody .... (1pm)
- ◯ **016** J. Pearson .... (2pm)
- ◯ **017** J. Boisvert .... (2pm)
- ◯ **018** R. Chestnut .... (4pm)
- ◯ **019** G. Surman .... (4pm)
- ◯ **020** N. Sagullo .... (8am)

Box your answers.

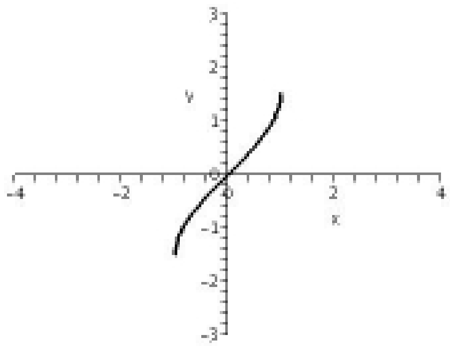
If you have a question raise your hand and remain seated. In order to receive full credit your answer must be **complete**, **legible** and **correct**. Show all of your work, and give adequate explanations. No calculators, no books, no notes are allowed on this exam.

DO NOT WRITE IN THIS BOX!

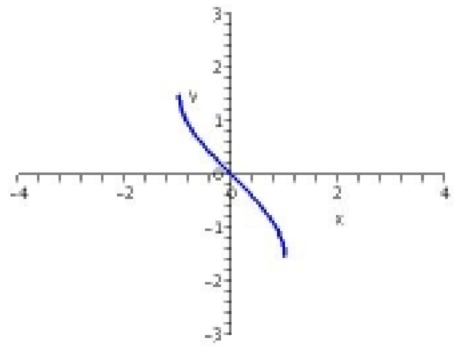
Problem	Points	Score
1	15 pts	
2	20 pts	
3	25 pts	
4	20 pts	
5	20 pts	
6	25 pts	
7	15 pts	
8	10 pts	
TOTAL	150 pts	

1. Sketch the graph of the function $f(x) = -\sin^{-1}\left(1+\dfrac{x}{2}\right)$ by translating, reflecting, compressing, stretching the graph of the inverse sine function.

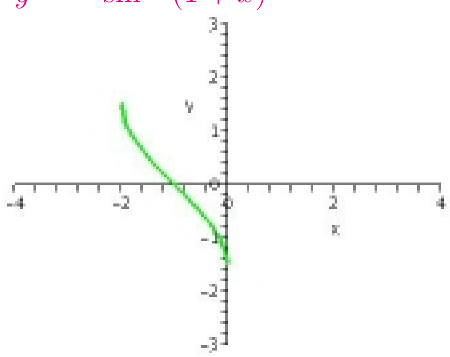
$y = \sin^{-1} x$



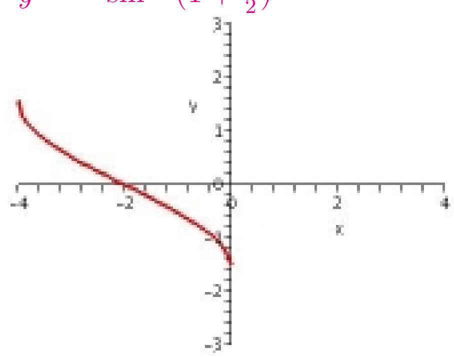
$y = -\sin^{-1} x$



$y = -\sin^{-1}(1+x)$



$y = -\sin^{-1}(1+\frac{x}{2})$



2. Let $f(x) = \ln(1-x)$ and $g(x) = \sqrt{x}$.

(a) Find a formula for the composite function $g \circ f$, and determine its domain.

$$(g \circ f)(x) = g(f(x)) = \sqrt{\ln(1-x)}\,.$$

$$\texttt{Domain:}\quad \{x : x < 1 \ \texttt{and}\ \ln(1-x) \geq 0\} = \{x : x < 1 \ \texttt{and}\ 1-x \geq 1\} = \{x : x \leq 0\} = (-\infty, 0]\,.$$

(b) Find a formula for the composite function $f \circ g$, and determine its domain.

$$(f \circ g)(x) = f(g(x)) = \ln(1-\sqrt{x})\,.$$

$$\texttt{Domain:}\quad \{x : x \geq 0 \ \texttt{and}\ 1-\sqrt{x} > 0\} = \{x : x \geq 0 \ \texttt{and}\ x < 1\} = [0, 1)\,.$$

3. Let f be the function with domain $[2, \infty)$ defined by $f(x) = x^2 - 2x - 3$.

(a) Explain why the function f has an inverse function f^{-1}.

To see that f has an inverse function we have to check that it is one-to-one.
f is one-to-one, because $f(x) = (x-1)^2 - 4$, $x \geq 2$, so f is increasing.

(b) Find the domain and the range of the inverse function f^{-1}.

Domain of f^{-1} = Range of f = $[-3, \infty)$.
Range of f^{-1} = Domain of f = $[2, \infty)$.

(c) Find a formula for f^{-1}.

$$y = x^2 - 2x - 3$$
$$0 = x^2 - 2x - (3 + y)$$
$$x = \frac{2 \pm \sqrt{4 + 4(3 + y)}}{2} = 1 \pm \sqrt{4 + y}$$

For $y \geq -3$ we must have $x \geq 2$, therefore $x = 1 + \sqrt{4 + y}$.
Hence $f^{-1}(y) = 1 + \sqrt{4 + y}$, $y \geq -3$,
or $f^{-1}(x) = 1 + \sqrt{4 + x}$, $x \geq -3$.

(d) Determine whether or not the function $g(x) = x^4 - 5x^2$ (with its natural domain) has an inverse function. Justify your answer. (You are **not** asked to find a formula for g^{-1}.)

g does not have an inverse function, because it is not one-to-one; for example, $g(-1) = g(1) = -4$.

(e) Determine whether or not the function $h(x) = \dfrac{x+1}{x-1}$ (with its natural domain) has an inverse function. Justify your answer. (You are **not** asked to find a formula for h^{-1}.)

h has an inverse function. To show this we have to check that h is one-to-one, that is, its graph passes the horizontal line test.
The graph of $h(x) = \frac{x+1}{x-1} = 1 + \frac{2}{x-1}$ is obtained from the graph of $y = \frac{1}{x}$ by vertical stretching and vertical and horizontal translations.
Since the graph of $y = \frac{1}{x}$ passes the horizontal line test, it follows that the graph of h also passes the horizontal line test.

4. Compute:

(a) $\cos\left(-\frac{2\pi}{3}\right) =$

$$-\frac{1}{2}.$$

(b) $\ln\left(\frac{1}{\sqrt{e}}\right) =$

$$\ln(e^{-1/2}) = -\frac{1}{2}.$$

(c) $e^{(\ln 5 + \ln \frac{1}{7})} =$

$$e^{\ln(5 \cdot \frac{1}{7})} = \frac{5}{7}.$$

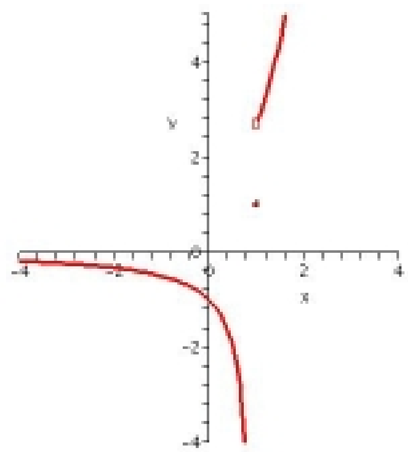
(d) $100 \log_2(\sqrt[50]{8}) =$

$$100 \log_2(8^{1/50}) = \frac{100}{50} \log_2(8) = \frac{100}{50} \cdot 3 = 6.$$

5. Let

$$f(x) = \begin{cases} e^x & \text{if } x > 1, \\ 1 & \text{if } x = 1, \\ \frac{1}{x-1} & \text{if } x < 1. \end{cases}$$

(a) Sketch the graph of f.



(b) Use the graph to find the following limits.
(The possible answers are: a number or ∞ or $-\infty$; write 'none' if none of these applies.)

$\lim_{x\to 1} f(x) =$

`none.`

$\lim_{x\to 1^+} f(x) =$

e.

$\lim_{x\to 1^-} f(x) =$

$-\infty$.

$\lim_{x\to \infty} f(x) =$

∞.

$\lim_{x\to -\infty} f(x) =$

0.

6. Compute the following limits.
(The possible answers are: a number or ∞ or $-\infty$; write 'none' if none of these applies.)

(a) $\lim\limits_{x\to 3} \dfrac{x^2-2x-3}{x^2-4x+3} =$

$$\lim_{x\to 3} \frac{(x+1)(x-3)}{(x-1)(x-3)} = \lim_{x\to 3} \frac{x+1}{x-1} = \frac{4}{2} = 2\,.$$

(b) $\lim\limits_{x\to 3^-} \dfrac{x^2-2x-3}{x^2-6x+9} =$

$$\lim_{x\to 3^-} \frac{(x+1)(x-3)}{(x-3)^2} = \lim_{x\to 3^-} \frac{x+1}{x-3} = \lim_{x\to 3^-} \left(1+\frac{4}{x-3}\right) = -\infty\,.$$

(c) $\lim\limits_{x\to 9} \dfrac{\sqrt{x}-3}{x-9} =$

$$\lim_{x\to 9} \frac{\sqrt{x}-3}{(\sqrt{x}-3)(\sqrt{x}+3)} = \lim_{x\to 9} \frac{1}{\sqrt{x}+3} = \frac{1}{6}\,.$$

(d) $\lim\limits_{x\to \infty} \dfrac{2x^3+3}{5x^3-4x^2+6} =$

$$\lim_{x\to \infty} \frac{2+\frac{3}{x^3}}{5-\frac{4}{x}+\frac{6}{x^3}} = \frac{2}{5}\,.$$

(e) $\lim\limits_{x\to -\infty} \dfrac{\sqrt{3x^2+1}-1}{2x+3} =$

$$\lim_{x\to -\infty} \frac{\frac{\sqrt{3x^2+1}}{-x}-\frac{1}{-x}}{\frac{2x}{-x}+\frac{3}{-x}} = \lim_{x\to -\infty} \frac{\sqrt{3+\frac{1}{x^2}}-\frac{1}{-x}}{-2+\frac{3}{-x}} = \frac{\sqrt{3}}{-2} = -\frac{\sqrt{3}}{2}\,.$$

7. TRUE or FALSE? Justify your answer.

(a) *If two functions f and g have the same domain, then f and g are equal.*

TRUE **[FALSE]** (circle one)

Justify:

For example, the functions $f(x) = x$ and $g(x) = x^2$ have the same domain $(-\infty, \infty)$, but they are different functions, since $f(2) \neq g(2)$.

(b) $\sin^{-1}(\sin x) = x$ for all real numbers x.

TRUE **[FALSE]** (circle one)

Justify:

For example, $\sin^{-1}(\sin \pi) = \sin^{-1} 0 = 0 \neq \pi$.

(c) *If a function $f(x)$ has a limit as x approaches 2, then the function*

$$g(x) = \begin{cases} 0 & \text{if } x \leq 1, \\ 3 & \text{if } x = 2, \\ f(x) & \text{for all other } x \text{ in the domain of } f \end{cases}$$

also has a limit as x approaches 2.

[TRUE] FALSE (circle one)

Justify:

$\lim_{x\to 2} f(x) = \lim_{x\to 2} g(x)$, since $f(x) = g(x)$ for all x such that $|x-2| < 1$, $x \neq 2$.

8. State the theorem on the relationship between one-sided and two-sided limits.

For arbitrary function f and real numbers a and L, $\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x)$.