Plymouth State UNIVERSITY

MA 2550: Calculus I

Partial Solutions to Even Numbered Exercises in Section 4.2

4. Let $f(x) = \cos 2x$. First, we verify that f satisfies all three of the hypotheses of Rolle's Theorem on the interval $[\pi/8, 7\pi/8]$.

   (1) Since $y = \cos x$ and $y = 2x$ are continuous and the composition of continuous functions is continuous, f is continuous on all of $\mathbb{R}$. This implies that f is continuous on $[\pi/8, 7\pi/8]$.

   (2) We see that (by the Chain Rule)

   $$f'(x) = -2\sin 2x.$$

   It is clear that f is differentiable on all of $\mathbb{R}$. This implies that f is differentiable on $(\pi/8, 7\pi/8)$.

   (3) We see that

   $$f(\pi/8) = \cos \pi/4 = \sqrt{2}/2$$

   and that

   $$f(7\pi/8) = \cos 7\pi/4 = \sqrt{2}/2.$$

   Therefore, $f(\pi/8) = f(7\pi/8)$.

   This verifies that f satisfies the three hypotheses of Rolle's Theorem. Now, we find all numbers c that satisfy the conclusion of Rolle's Theorem. The conclusion of Rolle's Theorem is that there exists at least one number c in the interval $(\pi/8, 7\pi/8)$ such that $f'(c) = 0$. We see that if $f'(x) = 0$, then

   $$\begin{aligned} 0 &= -2\sin 2x \\ 0 &= \sin 2x \\ 0 &= 2\sin x\cos x \qquad \text{(there's that double angle formula again)} \\ 0 &= \sin x \cos x. \end{aligned}$$

   By looking at the unit circle, we see that $\sin x = 0$ does not have a solution in the interval $(\pi/8, 7\pi/8)$. However, again by looking at the unit circle, the only value of x in the interval $(\pi/8, 7\pi/8)$ that satisfies $\cos x = 0$ is $x = \pi/2$. Therefore, $c = \pi/2$.

14. Let

   $$f(x) = \frac{x}{x+2}.$$

   First, we verify both conditions of the MVT.

   (1) Since rational functions are continuous on their domains, f is continuous everywhere, except $x = -2$. Therefore, f is continuous on $[1, 4]$.

(2) Since rational functions are differentiable on their domains, f is differentiable everywhere, except $x = -2$. Therefore, f is differentiable on $(1, 4)$.

This verifies that f satisfies the hypotheses of the MVT. Therefore, there exists at least one c in the interval $(1, 4)$ such that

$$f'(c) = \frac{f(4) - f(1)}{4 - 1} = \frac{2/3 - 1/3}{3} = \frac{1}{9}.$$

To find all c, we find the derivative and set it equal to 1/9. We see that (by the Quotient Rule)

$$f'(x) = \frac{2}{(x+2)^2}.$$

Then if

$$\frac{1}{9} = \frac{2}{(x+2)^2},$$

we find that $x = -2 \pm 3\sqrt{2}$. However, only $x = -2 + 3\sqrt{3}$ lies in the interval $(1, 4)$. Thus, $c = -2 + 3\sqrt{3}$.

32. This problem is tricky. We have to apply the MVT to the velocity function (first derivative), so that we can make a conclusion about acceleration (second derivative). Let $p(t)$ be the position function of the car at time t. Then p' is the velocity function of the car. Also, p' is continuous and differentiable everywhere (no worm holes, etc.). We need to get the units to match up. 10 minutes have elapsed from 2:00 PM to 2:10 PM. This is 1/6 of an hour. By the MVT, there exists at least one c in the interval $(0, 1/6)$ such that

$$p''(c) = \frac{p'(1/6) - p'(0)}{1/6 - 0} = \frac{50 - 30}{1/6} = 120.$$

This implies that there is at least one moment in time between 2:00 PM and 2:10 PM such that the acceleration was 120 mi/hr^2.