MA 3110.01: Logic, Proof and Axiomatic Systems

Solution to Exercise 8(c) from Section 1.3

Theorem. *Let $A(x)$ be an open sentence with variable x. Then*

$$(\exists! x)A(x)$$

is equivalent to

$$(\exists x)A(x) \wedge (\forall y)(\forall z)(A(y) \wedge A(z) \implies y = z).$$

To show that these two quantified sentences are equivalent, we need to show that their respective truth sets are equal in any universe. The "easiest" way to understand this proof is to do it in two halves. First, we'll show that if $(\exists! x)A(x)$ is true, then $(\exists x)A(x) \wedge (\forall y)(\forall z)(A(y) \wedge A(z) \implies y = z)$ is true. Second, we'll show the converse. That is, if $(\exists x)A(x) \wedge (\forall y)(\forall z)(A(y) \wedge A(z) \implies y = z)$ is true, then $(\exists! x)A(x)$ is true. This will show that $(\exists! x)A(x)$ is true iff $(\exists x)A(x) \wedge (\forall y)(\forall z)(A(y) \wedge A(z) \implies y = z)$ is true, which implies that both quantified sentences have the same truth set.

Before we get started, it is worth discussing what's intuitively going on with the quantified sentence $(\exists x)A(x) \wedge (\forall y)(\forall z)(A(y) \wedge A(z) \implies y = z)$. This sentence consists of two smaller sentences joined together by the "and" connective. The first part is $(\exists x)A(x)$, which says "there is at least one x such that $A(x)$." The second part is $(\forall y)(\forall z)(A(y) \wedge A(z) \implies y = z)$, which is harder to wrap your brain around. This bit basically says "there is at most one value in the universe that makes $A(x)$ true." To see that this is the case, think about what would have to happen in order for $A(y) \wedge A(z) \implies y = z$ to be false. We would need $A(y) \wedge A(z)$ be true (which happens iff both $A(y)$ and $A(z)$ are true), but $y = z$ to be false (or, in other words, $y \neq z$ would have to be true). So, for $A(y) \wedge A(z) \implies y = z$ to be true, it is NOT the case that $A(y)$ and $A(z)$ are true while $y \neq z$.

What follows is extremely verbose, but I wanted things to be as clear as possible. OK, let's do the proof.

Proof. Let U be any universe.

For the first half, assume that $(\exists! x)A(x)$ is true in U. This implies that there is a unique element, say x_0, in U such that $A(x)$ is true. Also, by Exercise 8(a), we know $(\exists x)A(x)$ is true, as well. Now, suppose that y and z are any two elements (maybe the same element) in U. If either of $A(y)$ or $A(z)$ is false, then $A(y) \wedge A(z)$ is false, which automatically makes $A(y) \wedge A(z) \implies y = z$ true. So, assume that both $A(y)$ and $A(z)$ are true. But since the truth set only has x_0 in it, it must be the case that $y = x_0$ and $z = x_0$, which clearly implies that $y = z$. Since we chose arbitrary y and z in U, this implies that $(\forall y)(\forall z)(A(y) \wedge A(z) \implies y = z)$ is true.

Since $(\exists x)A(x)$ and $(\forall y)(\forall z)(A(y) \wedge A(z) \implies y = z)$ are both true, it must be the case that $(\exists x)A(x) \wedge (\forall y)(\forall z)(A(y) \wedge A(z) \implies y = z)$ is true (assuming $(\exists! x)A(x)$ is true), as desired.

Now, for the second half, assume that $(\exists x)A(x) \wedge (\forall y)(\forall z)(A(y) \wedge A(z) \implies y = z)$ is true. Then $(\exists x)A(x)$ and $(\forall y)(\forall z)(A(y) \wedge A(z) \implies y = z)$ are both true. Since $(\exists x)A(x)$ is true, the truth set of $A(x)$ contains at least one element. Since $(\forall y)(\forall z)(A(y) \wedge A(z) \implies y = z)$ is true, the truth set for $A(y) \wedge A(z) \implies y = z$ is all pairs of y and z in U. But this implies that the truth set for $A(x)$ contains at most one element. Since we know that the truth set for $A(x)$ has at least one element and at most one element, it must be the case that it has exactly one element. Therefore, $(\exists! x)A(x)$ is true (assuming $(\exists x)A(x) \wedge (\forall y)(\forall z)(A(y) \wedge A(z) \implies y = z)$ is true).

$\square$