MA 3110: Logic, Proof, and Axiomatic Systems (Fall 2008)

EXAM 2 (Take-home portion)

NAME: SOLUTIONS

Instructions: These directions are slightly different than for Exam 1; so, **read this entire page**. Prove THREE of the following theorems, where *at least one* of theorems proven is Theorem 4 or Theorem 5. You may choose to prove both Theorem 4 and Theorem 5, but you don't have to. I expect your proofs to be well-written, neat, and organized. You should write in complete sentences. Do not turn in rough drafts. What you turn in should be the "polished" version of potentially several drafts.

This portion of Exam 2 is worth 30 points, where each proof is worth 10 points.

These are the simple rules for this portion of the exam:

1. You may freely use any theorems that we have discussed in class, but you should make it clear where you are using a previous result and which result you are using.
2. You are NOT allowed to copy someone else's work.
3. You are NOT allowed to let someone else copy your work.
4. You are allowed to discuss the problems with each other and critique each other's work.

If these simple rules are broken, then the remaining exams will be all in-class with no reduction in their difficulty.

This half of Exam 2 is due to my office (Hyde 312) by **5 pm on Friday, November 7th** (no exceptions). You should turn in this cover page and the three proofs that you have decided to submit.

Good luck and have fun!

Theorem 1: Let A, B, C be sets. If $A \subseteq B \cup C$ and $A \nsubseteq B$, then $A \cap B \neq \emptyset$.

Important: In its current form, this "theorem" is false. The conclusion was supposed to be $A \cap C \neq \emptyset$. You may either provide a counterexample to show that the original statement is false or you may prove the corrected statement.

The correct statement of the theorem is: If $A \subseteq B \cup C$ and $A \nsubseteq B$, then $A \cap C \neq \emptyset$. I will prove this.

Proof: Assume that $A \subseteq B \cup C$ and $A \nsubseteq B$. Since $A \nsubseteq B$, there exists $x \in A$ such that $x \notin B$. Since $A \subseteq B \cup C$, $x \in B \cup C$, which implies that $x \in B$ or $x \in C$. But $x \notin B$, and so it must be the case that $x \in C$. Then $x \in A \cap C$, which shows that $A \cap C \neq \emptyset$. $\square$

Theorem 2: Let A and B be sets in a universe U. Then $A \cup B^c = U$ iff $B \subseteq A$.

Proof: Let A and B be sets in a universe U.

($\Longrightarrow$) Assume $A \cup B^c = U$. We need to show that $B \subseteq A$. Let $x \in B$. Then $x \notin B^c$. Since $x \in U$ and $U = A \cup B^c$, $x \in A$ or $x \in B^c$. But $x \notin B^c$, and so it must be the case that $x \in A$. Thus, $B \subseteq A$.

($\Longleftarrow$) Assume $B \subseteq A$. We need to show that $A \cup B^c = U$. Since A and B^c are subsets of the universe U, $A \cup B^c \subseteq U$. It remains to show that $U \subseteq A \cup B^c$. Let $x \in U$. There are two possibilities: either $x \in B$ or $x \notin B$. If $x \in B$, then $x \in A$ since $B \subseteq A$. In this case, x would be an element of $A \cup B^c$, as desired. On the other hand, if $x \notin B$, then $x \in B^c$, in which case $x \in A \cup B^c$. So, in either case, we have $x \in A \cup B^c$. Therefore, $U \subseteq A \cup B^c$. We have shown that $U = A \cup B^c$. $\square$

Theorem 3: Let A, B, C be sets. If $C \subseteq A \cap B$, then $A \subseteq (A - C) \cup B$.

Proof: Assume that $C \subseteq A \cap B$. We need to show that $A \subseteq (A - C) \cup B$. Let $x \in A$. There are two possibilities: $x \in C$ or $x \notin C$. In the first case, if $x \in C$, then $x \in A \cap B$, which implies that $x \in B$. But then $x \in (A - C) \cup B$, as desired. In the second case, if $x \notin C$, then it must be the case that $x \in A - C$ (since $x \in A$ but $x \notin C$). In this case, we again have $x \in (A - C) \cup B$. So, in either case, $x \in (A - C) \cup B$. Therefore, $A \subseteq (A - C) \cup B$. $\square$

Theorem 4: For every $n \in \mathbb{N}$, $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - \frac{1}{n}$.

Proof: We proceed by induction.

(i) Basic Step: We see that

$$\sum_{i=1}^{1} \frac{1}{i^2} = \frac{1}{1^2} = 1,$$

and, on the other hand,

$$2 - \frac{1}{1} = 1.$$

This shows that the statement is true when $n = 1$.

(ii) Inductive Step: Let $n \in \mathbb{N}$ and assume that

$$\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - \frac{1}{n}.$$

We need to show that

$$\sum_{i=1}^{n+1} \frac{1}{i^2} \leq 2 - \frac{1}{n+1}.$$

We see that

$$\begin{aligned}
\sum_{i=1}^{n+1} \frac{1}{i^2} &= \sum_{i=1}^{n} \frac{1}{i^2} + \frac{1}{(n+1)^2} \\
&\leq 2 - \frac{1}{n} + \frac{1}{(n+1)^2} && \text{(by induction hypothesis)} \\
&= 2 - \frac{(n+1)^2 - n}{n(n+1)^2} \\
&= 2 - \frac{n^2 + n + 1}{n(n+1)^2} \\
&\leq 2 - \frac{n^2 + n + 1}{n(n+1)^2} + \frac{1}{n(n+1)^2} && \text{(added positive number)} \\
&= 2 - \frac{n^2 + n}{n(n+1)^2} \\
&= 2 - \frac{1}{n+1}.
\end{aligned}$$

(iii) By the PMI, the truth set is all of $\mathbb{N}$. This implies that for all $n \in \mathbb{N}$,

$$\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - \frac{1}{n}.$$

□

For the next theorem, you must prove both parts. Prove part (a) by induction.

Theorem 5: Let $\{A_n : n \in \mathbb{N}\}$ be a family of sets satisfying $A_n \subseteq A_{n+1}$ for all $n \geq 1$.

(a) For all $n \in \mathbb{N}$, $A_1 \subseteq A_n$.

Proof: Assume that $A_n \subseteq A_{n+1}$. We proceed by induction.

(i) Basic Step: We see that $A_1 \subseteq A_1$. So, statement is true for $n = 1$.

(ii) Inductive Step: Let $n \in \mathbb{N}$ and assume that $A_1 \subseteq A_n$. Since $A_1 \subseteq A_n$ and $A_n \subseteq A_{n+1}$, $A_1 \subseteq A_{n+1}$ (by transitivity of $\subseteq$; see Theorem 2.2).

(iii) By the PMI, the truth set is all of $\mathbb{N}$. So, $A_1 \subseteq A_n$ for every $n \in \mathbb{N}$. $\square$

(b) $\displaystyle\bigcap_{n=1}^{\infty} A_n = A_1$.

Proof: To prove that these two sets are equal, we will prove both containments separately.

($\subseteq$) Let $x \in \displaystyle\bigcap_{n=1}^{\infty} A_n$. Then $x \in A_n$ for every $n \in \mathbb{N}$. In particular, $x \in A_1$, which implies that $\displaystyle\bigcap_{n=1}^{\infty} A_n \subseteq A_1$.

($\supseteq$) Let $x \in A_1$. Then by part (a), $x \in A_n$ for all $n \in \mathbb{N}$. So, $x \in \bigcap_{n=1}^{\infty} A_n$. Hence $A_1 \subseteq \displaystyle\bigcap_{n=1}^{\infty} A_n$.

Therefore, $\displaystyle\bigcap_{n=1}^{\infty} A_n = A_1$. $\square$