Plymouth State UNIVERSITY

MA 3110: Logic, Proof, and Axiomatic Systems

Section 4.1: Functions as Relations (part 1)

NAME(S):

The goal of this handout is for you to teach yourself the concepts introduced in the first half of Section 4.1. My intention is that you will form small groups and work your way through the next few pages together. Feel free to use the book to supplement this handout.

I want you to turn in this handout on **Monday, November 24**. Up to 3 people can put their names on a single handout. For example, if 3 people (or less) work together, then they can all put their names on a single handout and just turn in that one. If 4 (or more) people work together, then you'll have to turn in (at least) 2 handouts. You may work on this alone, but I'd prefer if you didn't.

There are missing words (indicated by underlined blanks) that you will need to fill in along the way. You ought to be able to fill in the missing words by either recalling ideas from earlier sections or inferring the missing words by understanding what comes earlier in the handout. OK, let's get started!

Recall: A *relation* R from a set A to a set B is a collection of ____________ from the set $A \times B$. That is, $R \subseteq A \times B$. We have seen many different ways of representing relations:

1. explicit list of ordered pairs;
2. a digraph;
3. a "bubble" diagram with arrows going from left to right (for an example of what I mean by this, see Figure 3.10 on page 140);
4. a graph in the xy-plane (which really is a list of ordered pairs);
5. a verbal description of what is related to what.

For this section, it will be most useful for us to deal with "bubble" diagrams and graphs in the xy-plane. Let's work out a few examples. (Note: In this section, we are not requiring that our relations be equivalence relations.)

Example 1: Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{w, x, y, z\}$. Let S be the relation from A to B given by

$$S = \{(1, x), (2, x), (1, y), (3, w), (5, y)\}.$$

Draw the "bubble" diagram that corresponds to S.

Example 2: Let A and B be as in Example 1. Let T be the relation from A to B given by

$$T = \{(1, x), (2, x), (4, y), (3, w), (5, y)\}.$$

Draw the "bubble" diagram that corresponds to T.

Example 3: Let C be the relation from $[-1, 1]$ to $\mathbb{R}$ given by

$$xCy \text{ iff } x^2 + y^2 = 1.$$

Draw the graph in the xy-plane that corresponds to C.

In this case, the graph of C is a _____________.

Example 4: Let P be the relation from $\mathbb{R}$ to $\mathbb{R}$ given by

$$xPy \text{ iff } y = x^2.$$

Draw the graph in the xy-plane that corresponds to P.

In this case, the graph of P is a _____________.

We will refer back to each of these examples throughout the rest of this handout.

Recall: The *domain* of a relation R from A to B is the set $\{$_____________$\}$. It is always the case that

$$Dom(R) \subseteq \text{\_\_\_\_\_\_\_\_\_\_\_\_\_},$$

and the domain of R need not be all of _____________.

Also, the *range* of a relation R from A to B is the set $\{$_____________$\}$. It is always the case that

$$Rng(R) \subseteq \text{\_\_\_\_\_\_\_\_\_\_\_\_\_},$$

and the range of R need not be all of _____________.

Example 5: Determine the domain and range of each of the relations S, T, C, and P from the previous examples.

We're now ready to define what a function is.

Definition: A *function* (or *mapping*) *from A to B* is a relation f from A to B such that

(i) the domain of f is all of A;

(ii) if $(x, y) \in f$ and $(x, z) \in f$, then $y = z$.

We write $f : A \to B$ and say "f is a function from A to B," or "f maps A to B." The set B is called the *codomain* of f. In the special case where $A = B$, we say that$f$ is a *function on* A.

Part (i) of the definition above is requiring that every element of A be related to something in B. So, in our "bubble" diagrams, every point in the left "bubble" must have an arrow leaving it. Part (ii) of the definition is requiring that every element of the domain have exactly _____________ element of the range related to it. So, in our "bubble" diagrams, at most _____________ arrow may leave a point in the domain.

Important: Some relations are functions and some are not. Since the definition of a function has two requirements, there are two ways in which a relation could fail to be a function: If R is a relation from A to B, then

1. if $Dom(R) \neq$ _____________, then R is not a function;
2. if there exists x in the domain of R such that x is related to (or mapped to) two different elements in the range, then R is not a function.

Missile Analogy: Here is a useful analogy. Think of the domain of a function as a set of missiles and the codomain as a set of targets. A function is a description of which missiles blow up which targets. In this case, a missile can hit at most one target. However, it is possible that more than one missile may blow up the same target. Furthermore, not every target may get hit. In this case, the targets that get blown up form the range (which may be smaller than the _____________).

Example 6: There are two reasons why the relation S from Example 1 is *not* a function.

1. First, we see that ______________ $\neq A$. This violates condition (i) of the definition of function.
2. Second, we see that the point ______________ in the domain of S is related to ______________ elements of the range. This violates condition (ii) of the definition.

Example 7: Explain why the relation T from Example 2 is a function. (There are 2 things that need to be said.) Identify the domain, codomain, and range for this function.

Example 8: The relation C from Example 3 is *not* a function. Give two examples of values in the domain of C that have more than one value related to them in the range.

Everyone should be familiar with the *vertical line test* that tests whether a graph is a function. Well, the vertical line test is applicable in this context, as well. If some vertical line hits the graph (not talking about digraph) of a relation R in more than one spot, then that means that there is a value x in the domain of R such that x is related to more than one value in the range. In this case, R is not a function. Notice that in the previous example, the graph of C fails the vertical line test.

Example 9: The relation P from Example 4 is a well-know function. In this case, the domain of P is $\mathbb{R}$. Each element $x \in Dom(P)$ has exactly one element y that it is related to. For example, the ordered pair $(2,$ ____$) \in P$ and there is no other ordered pair in P that has first coordinate 2. The codomain of P is ______________, but the range of P is ______________. Notice that for most elements in the range, there is more than one element of the domain that is related to it. For example, (____, 4) and (____, 4) are both elements of P. Again, a missile cannot blow up more than one target, but more than one missile can blow up the same target.

Example 10: Let P be the set of all people and let N be the set of first names. Then there is a relation R_1 from P to N. An element of R_1 is an ordered pair where the first coordinate is a person and the second coordinate is their first name. Explain why R_1 is a function. Is it okay that there is more than one person named Sally?

There is also a relation R_2 from N to P. In this case, an element of R_2 is an ordered pair where the first coordinate is a name and the second coordinate is a person. Explain why R_2 is *not* a function.

Definition: Let $f : A \to B$ (that is, f is a function from A to B). We write $y = f(x)$ when $(x, y) \in f$. We say that y is the *image of x under f* (or *value of f at x*) and that x is a *pre-image of y under f*.

Note: Notice that we said *the* image and *a* pre-image. The reason for this is that there can only be a single image for a point in the domain, but there may be many pre-images for a point in the range.

Example 11: Consider the function T from Example 2. Since $(4, y) \in T$, we write $T(4) = y$. In this case, we say that y is the image of 4 under T. Also, we say that 4 is a pre-image of y under T. However, ___ is also a pre-image of y. Similarly, since $(1, x) \in T$, $T($___$) =$ ___ and we say that ___ is the image of ___ under T and ___ and ___ are the pre-images of ___ under T.

Important: If you have a "bubble" diagram for a function (like we do for the previous example), pre-images live in the _____________ bubble while images live in the _____________ bubble.

Example 12: Rephrase the definitions of image and pre-image in terms of the missile analogy.

Example 13: Consider the function P from Example 4. We see that $P(2) =$ ___. The image of 0 under P is ___. The image of -3 under P is ___. The pre-image of 25 is ___ and ___.

Now, on your own or in the same group, complete the following exercises.

Section 4.1: # 1abg, 2ad, 4abc, 6a

You can try to squeeze these problems on here, staple them to this handout, or turn them in separately if you don't get to complete these problems in your group.