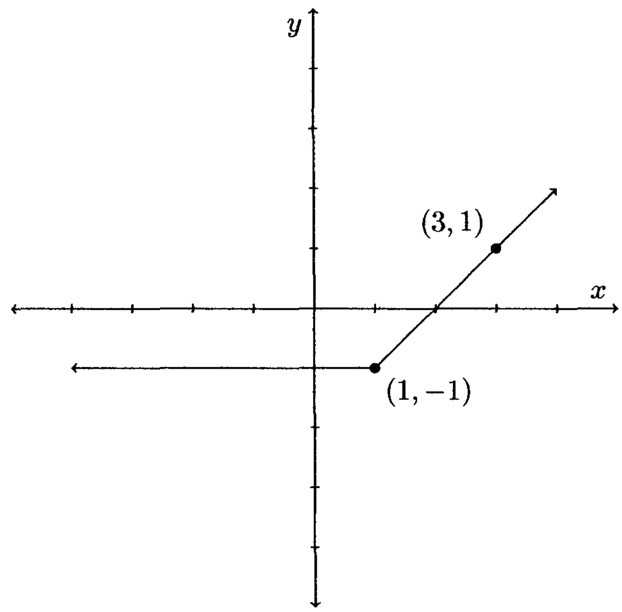
MA 2550: Calculus I (Fall 2009)
Exam 1

NAME: Solutions

Instructions: Answer each of the following questions completely. To receive full credit, you must *justify* each of your answers (unless stated otherwise). How you reached your answer is more important than the answer itself. If something is unclear, or if you have any questions, then please ask. Good luck!

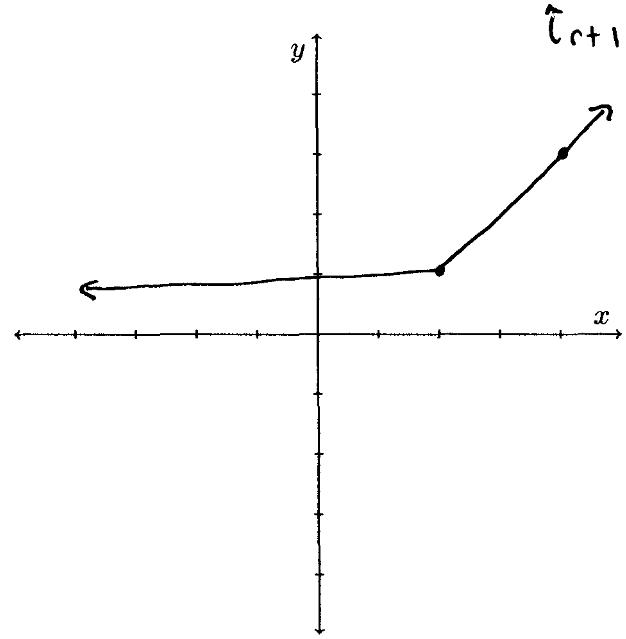
1. (6 points) Suppose the graph of a function $y = f(x)$ looks like:



Using the axes provided, sketch the graph of the function $y = f(x-1) + 2$.

up 2

r+1



2. (3 points each) Let a, b, c and d be real numbers. Match each function with the only possible correct graph.

(a) $f(x) = \dfrac{\cancel{(x-a)}\cancel{(x-b)}}{\cancel{(x-a)}\cancel{(x-b)}}$ Graph: C

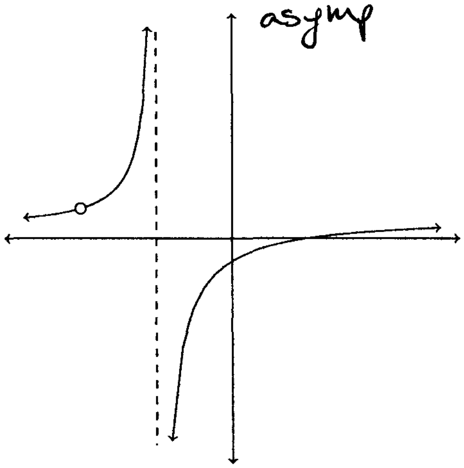
2 holes

(b) $g(x) = \dfrac{\cancel{(x-a)}(x-b)}{\cancel{(x-a)}(x-c)}$ Graph: A

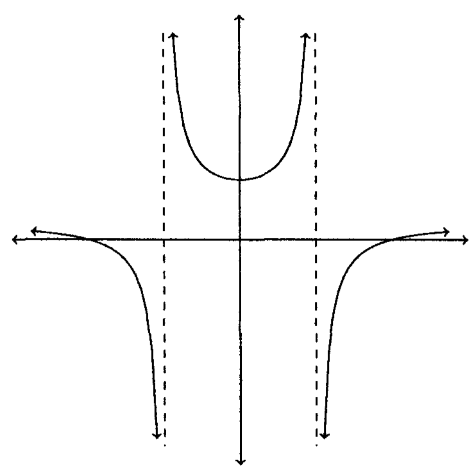
hole vert asymp

(c) $h(x) = \dfrac{(x-a)(x-b)}{(x-c)(x-d)}$ Graph: B

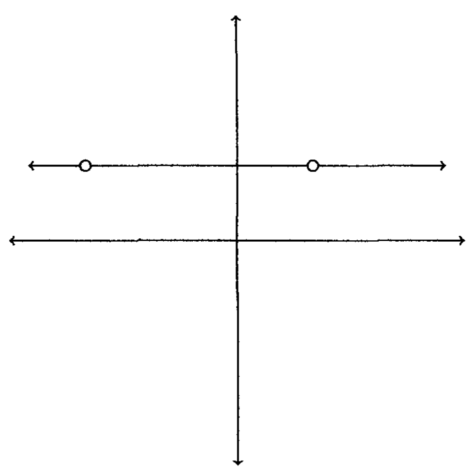
2 vert asymp



(a) Graph A



(b) Graph B



(c) Graph C

3. (3 points each) Consider the following function.

$$f(x) = \begin{cases} \dfrac{-1}{x+1}, & x \leq 2 \\ \sqrt{x}, & x > 2 \end{cases}$$

For (a)–(h), evaluate the given expression. If an expression does not exist, specify whether it equals ∞, $-\infty$, or simply does not exist (in which case, write DNE). You do *not* need to justify your answers.

(a) $\lim_{x\to -1^-} f(x) = \boxed{\infty}$

(b) $\lim_{x\to -1^+} f(x) = \boxed{-\infty}$

(c) $\lim_{x\to -1} f(x)$ $\boxed{\text{DNE}}$

(d) $f(-1)$ $\boxed{\text{DNE}}$

(e) $\lim_{x\to 2^-} f(x) = \boxed{-\frac{1}{3}}$

(f) $\lim_{x\to 2^+} f(x) = \boxed{\sqrt{2}}$

(g) $\lim_{x\to 2} f(x)$ $\boxed{\text{DNE}}$

(h) $f(2) = \boxed{-\frac{1}{3}}$

(i) Identify any x-values where f has discontinuities.

f has discont @ $x = -1, 2$

4. (8 points each) Evaluate each of the following limits. If a limit does not exist, specify whether the limit equals ∞, $-\infty$, or simply does not exist (in which case, write DNE). Sufficient work must be shown and proper notation should be used. Give *exact answers.*

(a) $\lim_{x\to -2} \dfrac{x^2+2x}{x^2-4}$ $\left(\frac{0}{0}\right)$

$$= \lim_{x\to -2} \frac{x\cancel{(x+2)}}{(x-2)\cancel{(x+2)}}$$

$$= \lim_{x\to -2} \frac{x}{x-2}$$

$$= \frac{-2}{-4}$$

$$= \boxed{\frac{1}{2}}.$$

(b) $\lim_{x\to 9} \dfrac{\sqrt{x}-3}{x-9}$ $\left(\frac{0}{0}\right)$

$$= \lim_{x\to 9} \frac{\sqrt{x}+3}{\sqrt{x}+3} \cdot \frac{\sqrt{x}-3}{x-9}$$

$$= \lim_{x\to 9} \frac{\cancel{x-9}}{(\sqrt{x}+3)\cancel{(x-9)}}$$

$$= \lim_{x\to 9} \frac{1}{\sqrt{x}+3}$$

$$= \frac{1}{\sqrt{9}+3} = \boxed{\frac{1}{6}}$$

(c) $\lim_{x\to 0} \frac{\frac{1}{x-2}+\frac{1}{2}}{x}$ $\left(\frac{0}{0}\right)$

$$= \lim_{x\to 0} \frac{2+x-2}{2(x-2)} \cdot \frac{1}{x}$$

$$= \lim_{x\to 0} \frac{\cancel{x}}{2(x-2)} \cdot \frac{1}{\cancel{x}}$$

$$= \frac{1}{2(0-2)}$$

$$= \boxed{-\frac{1}{4}}$$

(d) $\lim_{x\to \pi} \frac{\sin x}{x}$

$$= \frac{\sin \pi}{\pi}$$

$$= \frac{0}{\pi}$$

$$= \boxed{0}$$

5. (6 points) Use the Squeeze Theorem to evaluate $\lim_{x\to 0} x^2 \sin\left(\frac{17}{x}\right)$.

Note that $-1 \le \sin(17/x) \le 1$. Then

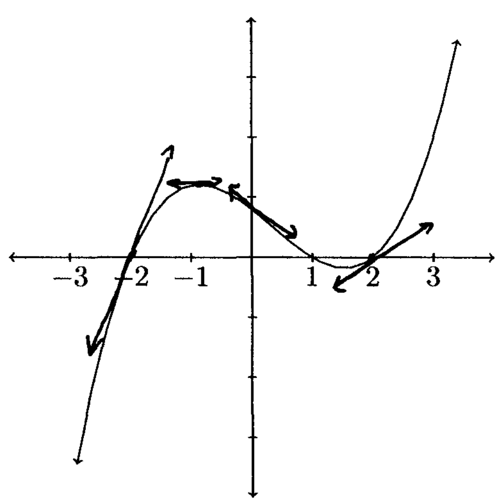
$-x^2 \le x^2 \sin(17/x) \le x^2$. Also,

$$\lim_{x\to 0} -x^2 = 0 = \lim_{x\to 0} x^2.$$

Thus, by Squeeze thm,

$$\lim_{x\to 0} x^2 \sin(17/x) = \boxed{0}.$$

6. (6 points) Consider the graph of the function $y = f(x)$ given below.



Put the following expressions in increasing order: $f'(-2), f'(-1), f'(0), f'(2)$.

small ⟶ big

$f'(0),\ f'(-1),\ f'(2),\ f'(-2)$

7. (8 points) The function $f(x) = \dfrac{x+1}{x-3}$ has the following derivative at $x = a$:

$$f'(a) = \frac{-4}{(a-3)^2}$$

(You do *not* need to show this.) Find the equation of the tangent line to the graph of f at $a = 1$. (It does not matter what form the equation of your line takes.)

We will use $y - y_0 = m(x - x_0)$, where

$x_0 = a = 1$, $y_0 = f(a) = f(1) = -1$, and

$m = f'(a) = f'(1) = -4/(1-3)^2 = -1$. So, eqn

of tangent line to graph of f @ $x=1$

is $\boxed{y + 1 = -1 \cdot (x - 1)}$ (or $y = -x$)

8. (6 points) Achilles and Tortoise decide to race up Zeno Mountain by two different paths. Because Achilles is so much faster, he decides that he will take the longer of the two paths and give Tortoise an hour head start. Achilles has the ability to move much faster than Tortoise, but because he is arrogant and easily distracted, he often loses races like this to Tortoise. However, this time, Achilles reaches the summit of Zeno Mountain just ahead of Tortoise exactly 5 hours after Tortoise started up his path. Use the Intermediate Value Theorem to argue that there is at least one moment in time when Achilles and Tortoise have the same elevation on their respective paths during the time interval that Achilles is moving up the mountain. (Assume that both Achilles and Tortoise are never going downhill on their way up their respective paths.)

(Hint: let $h_A(t)$ be the elevation Achilles after time t and let $h_T(t)$ be the elevation for Tortoise after time t. You need to argue that there exists a $c \in (1,5)$ such that $h_A(c) = h_T(c)$. Use the IVT on the function $y = h_A(t) - h_T(t)$.)

(Assuming Tortoise never went downhill)

(1) $h_A(t)$ and $h_T(t)$ are cont.

(2) So, $y = h_A(t) - h_T(t)$ is cont.

(3) $h_A(1) - h_T(1) = 0 - (\text{pos } \#) = \text{neg } \#$, and

$h_A(5) - h_T(5) = \text{top} - (\text{not top}) = \text{pos } \#$.

(4) By IVT, there exists $c \in (1,5)$ s.t.

$h_A(c) - h_T(c) = 0 \iff h_A(c) = h_T(c)$.

9. **Bonus Question:** (5 points, missing this question will not count against you) Using the limit definition of the derivative, prove that $f'(0)$ does not exist when $f(x) = |x|$.

If $f'(0)$ exists, then

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{|0+h| - |0|}{h}$$

$$= \lim_{h \to 0} \frac{|h|}{h},$$

but $\displaystyle\lim_{h \to 0^+} \frac{|h|}{h} = 1$ while

$\displaystyle\lim_{h \to 0^-} \frac{|h|}{h} = -1$. So, $\displaystyle\lim_{h \to 0} \frac{|h|}{h}$ DNE,

which implies that $f'(0)$ DNE.