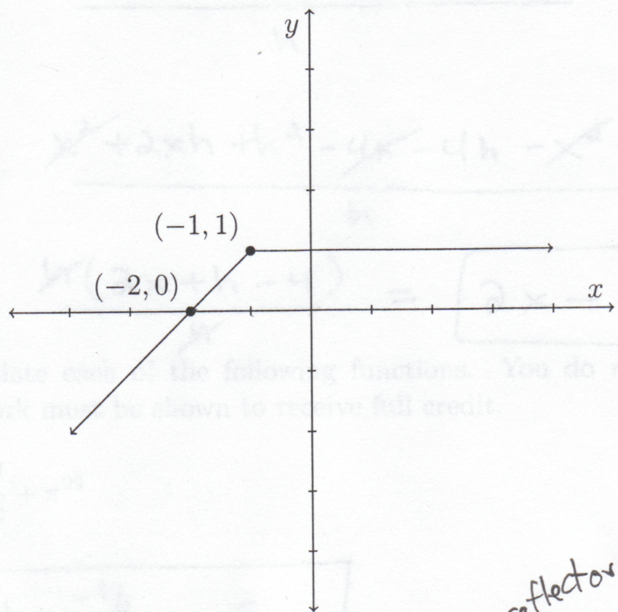
MA 2550: Calculus I (Fall 2009)
Exam 2

NAME: Solutions

Instructions: Answer each of the following questions completely. To receive full credit, you must *justify* each of your answers (unless stated otherwise). How you reached your answer is more important than the answer itself. If something is unclear, or if you have any questions, then please ask. Good luck!

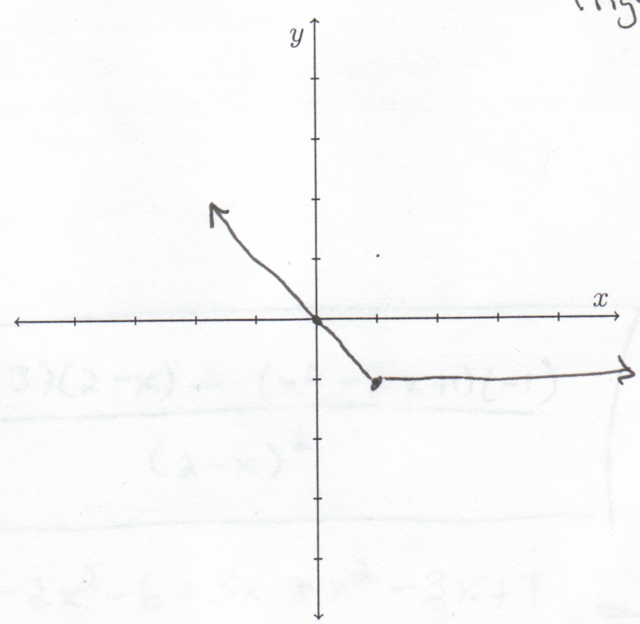
1. (6 points) Suppose the graph of a function $y = f(x)$ looks like:



Using the axes provided, sketch the graph of the function $y = -f(x-2)$.

reflect over x-axis

right 2



2. (8 points) Using the *limit definition of the derivative*, find the derivative of the following function. (No credit will be given for finding the derivative using another method.)

$$f(x) = x^2 - 4x$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 - 4(x+h) - (x^2 - 4x)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 \cancel{-4x} - 4h \cancel{-x^2} \cancel{+4x}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(2x + h - 4)}{\cancel{h}} = \boxed{2x - 4}$$

3. (6 points each) Differentiate each of the following functions. You do *not* need to simplify your answers, but sufficient work must be shown to receive full credit.

(a) $f(x) = \frac{x^2}{2} + \sqrt{x} - \frac{3}{x} + \pi^{99}$

$$f'(x) = \boxed{x + \frac{1}{2}x^{-1/2} + \frac{3}{x^2}}$$

(b) $y = \frac{x^2 - 3x + 1}{2 - x}$

$$\frac{dy}{dx} = \boxed{\frac{(2x-3)(2-x) - (x^2 - 3x + 1)(-1)}{(2-x)^2}}$$

or

$$= \frac{4x - 2x^2 - 6 + 3x + x^2 - 3x + 1}{(2-x)^2} = \boxed{\frac{-x^2 + 4x - 5}{(2-x)^2}}$$

(c) $g(x) = \sec \dfrac{x}{5}$

$$g'(x) = \boxed{\frac{1}{5}\sec\left(\frac{x}{5}\right)\tan\left(\frac{x}{5}\right)}$$

4. (8 points) Find $\dfrac{dy}{dx}$ if $x^2y + y^2 = x$. You do *not* need to simplify your answer, but you do need to solve for $\dfrac{dy}{dx}$.

$$\frac{d}{dx}\left[x^2y + y^2\right] = \frac{d}{dx}\left[x\right]$$

$$2xy + x^2\frac{dy}{dx} + 2y\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \boxed{\frac{1-2xy}{x^2+2y}}$$

5. (8 points) Find the *equation* of the tangent line to the graph of $f(x) = \cos x$ at $x = \pi/6$. It does not matter what form the equation takes, but all coefficients should be simplified and have exact values (i.e., no decimal approximations).

$$y - y_0 = m(x - x_0)$$

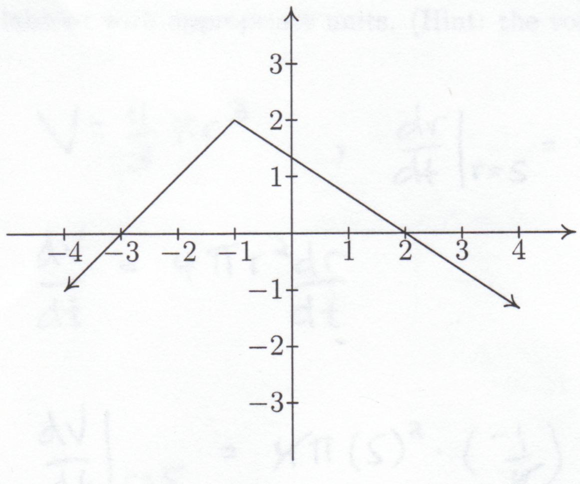
$$x_0 = \pi/6$$

$$y_0 = f(\pi/6) = \cos(\pi/6) = \frac{\sqrt{3}}{2}$$

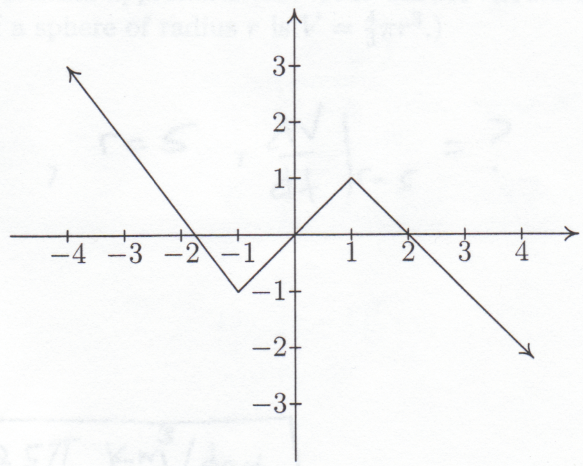
$$m = f'(\pi/6) = -\sin(\pi/6) = \frac{-1}{2}$$

So, eqn of line is $\boxed{y - \frac{\sqrt{3}}{2} = \frac{-1}{2}(x - \pi/6)}$

6. (4 points each) Consider the following graphs for functions f and g. Using the graphs, evaluate each of the following expressions. If an expression does not exist, write DNE.



Graph of f



Graph of g

(a) Find $g(0)$.

$\boxed{0}$

(b) Find $f'(0)$.

$\boxed{-\frac{2}{3}}$

(c) Find $f'(-1)$.

$\boxed{\text{DNE}}$

(d) Find $g'(0)$.

$\boxed{1}$

(e) Suppose $h(x) = f(g(x))$. Find $h'(0)$.

$h'(0) = f'(g(0))\, g'(0)$

$= f'(0) \cdot 1 = \boxed{-2/3}$

7. (8 points) A large spherical nugget is speeding towards Earth. If the radius of the nugget is decreasing at a rate of 1/4 kilometer per day, what is the rate of change in the volume of the nugget when the radius is 5 kilometers? Give an *exact* answer, not a decimal approximation. Your answer should be labeled with appropriate units. (Hint: the volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$.)

$V = \frac{4}{3}\pi r^3$, $\left.\frac{dr}{dt}\right|_{r=5} = -1/4$, $r = 5$, $\left.\frac{dV}{dt}\right|_{r=5} = ?$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\left.\frac{dV}{dt}\right|_{r=5} = \cancel{4}\pi(5)^2 \cdot \left(\frac{-1}{\cancel{4}}\right) = \boxed{-25\pi \text{ km}^3/\text{day}}$$

8. (8 points) Find all critical numbers of the following function.

$$f(x) = 5x^{2/3} + x^{5/3}$$

$$f'(x) = \frac{10}{3}x^{-1/3} + \frac{5}{3}x^{2/3}$$

$$0 = \frac{10 + 5x}{3x^{1/3}}$$

$$\boxed{\begin{array}{c} x = -2 \\ x = 0 \end{array}}$$

9. (8 points) Use appropriate calculus techniques to find the absolute maximum and absolute minimum values of the function $f(x) = x^3 - 3x^2 - 9x + 5$ on the interval $[1, 4]$. Sufficient work must be shown.

$f'(x) = 3x^2 - 6x - 9$

$0 = 3(x^2 - 2x - 3)$

$= 3(x-3)(x+1)$

$x = 3$ | $\cancel{x = -1}$

↑ not in $(1, 4)$

x	$f(x)$	
1	-6	— max
3	-22	— min
4	-15	

10. (8 points) Let

$$f(x) = 10 - \frac{16}{x}.$$

Show that f satisfies the hypotheses of the Mean Value Theorem on the interval $[2, 8]$, and then find all the numbers c that the Mean Value Theorem guarantees exist.

$f'(x) = \frac{16}{x^2}$

$\frac{f(8) - f(2)}{8 - 2}$

$= \frac{8 - 2}{8 - 2}$

$= 1$

$\frac{16}{x^2} = 1$

$x = \pm 4$,

except $-4 \notin (2, 8)$. So,

$\boxed{c = 4}$

f cont on $[2, 8]$

f diff on $(2, 8)$

11. **Bonus Question:** (5 points) A truck driver handed in a ticket at a toll booth showing that in 2 hours he had covered 158 mi on a toll road with speed limit 70 mph. The driver was cited for speeding. Use the Mean Value Theorem to explain why. Be sure to state the assumptions that we have to make about the position function $p(t)$ of the truck to be able to apply the Mean Value Theorem.

Let $p(t)$ be driver's pos fcn. Then $p(t)$ is cont on $[0,2]$ and diff on $(0,2)$.

By MVT, there exists $c \in (0,2)$ s.t.

$$p'(c) = \frac{p(2)-p(0)}{2-0}$$

$$= \frac{158-0}{2-0}$$

$$= 79 \text{ mph.}$$

We see that there was @ least one moment in time when driver had a velocity of 79 mph.