Solutions to Exercises 21 and 23 in Section 5.2

Here are some solutions to a couple of the more difficult exercises in Section 5.2.

21. Let $f(x) = 1 + 3x$. We are supposed to compute the definite integral of f from -1 to 5. Since f is continuous, we can take each x_i^* to be the right endpoint of the ith subinterval. That is, we can take $x_i^* = x_i$. To compute this integral, we need to find a few pieces of information. We see that

$$\Delta x = \frac{b-a}{n} = \frac{5-(-1)}{n} = \frac{6}{n}$$

and

$$x_i = a + i\Delta x = -1 + \frac{6i}{n},$$

which implies that

$$f(x_i) = 1 + 3\left(-1 + \frac{6i}{n}\right).$$

Then

$$\begin{aligned}
\int_{-1}^{5} 1 + 3x \, dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(1 + 3\left(-1 + \frac{6i}{n}\right)\right)\frac{6}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{-12}{n} + \frac{108i}{n^2}\right) \\
&= \lim_{n\to\infty} \left(\sum_{i=1}^{n} \frac{-12}{n} + \sum_{i=1}^{n} \frac{108i}{n^2}\right) \\
&= \lim_{n\to\infty} \left(\frac{-12}{n}\sum_{i=1}^{n} 1 + \frac{108}{n^2}\sum_{i=1}^{n} i\right) \\
&= \lim_{n\to\infty} \left(\frac{-12}{n}\cdot n + \frac{108}{n^2}\cdot\frac{n(n+1)}{2}\right) \\
&= -12 + 54 \\
&= \boxed{42}.
\end{aligned}$$

A few comments are in order. All that happened in going from the second line to the third line above is that everything got multiplied out and like terms were combined. In the next step we split the sum into two sums. We can do this since addition is commutative (order does not matter). Next, we factored out everything that didn't have an i in it. After that we replaced both sums with what they are equal to. The sum on the right got replaced with one of the useful sum formulas. The left sum is telling us to add 1 to itself n times, which is equal to n. That is,

$$\sum_{i=1}^{n} 1 = \underbrace{1 + 1 + \cdots 1}_{n \text{ times}} = n.$$

Lastly, we applied our shortcuts for evaluating infinite limits of rational functions. Here, the numerator and the denominator have equal degree, so we get the ratio of the leading coefficients in each case.

23. Let $f(x) = 2 - x^2$. Again, this function is continuous. So, we can use right endpoints again. We see that

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n}$$

and

$$x_i = a + i\Delta x = 0 + \frac{2i}{n} = \frac{2i}{n},$$

which implies that

$$f(x_i) = 2 - \left(\frac{2i}{n}\right)^2 = 2 - \frac{4i^2}{n^2}.$$

Then

$$\begin{aligned}
\int_0^2 2 +^2 x \ dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(2 - \frac{4i^2}{n^2}\right)\frac{2}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{4}{n} - \frac{8i^2}{n^3}\right) \\
&= \lim_{n\to\infty} \left(\sum_{i=1}^{n} \frac{4}{n} - \sum_{i=1}^{n} \frac{8i^2}{n^3}\right) \\
&= \lim_{n\to\infty} \left(\frac{4}{n}\sum_{i=1}^{n} 1 - \frac{8}{n^3}\sum_{i=1}^{n} i^2\right) \\
&= \lim_{n\to\infty} \left(\frac{4}{n}\cdot n - \frac{8}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}\right) \\
&= 4 - \frac{8\cdot 2}{6} \\
&= \boxed{4/3}.
\end{aligned}$$