Section 12.10: Taylor & Maclaurin Series

Goal

We will introduce Taylor and Maclaurin Series, which are special types of power series.

Initial discussion

Recall 1. A *power series* is an infinite power series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \cdots$$

We say that the power series is centered at x = a.

Question 2. What functions have power series representations?

Example 3. Consider the function

$$f(x) = \frac{1}{1-x}.$$

Have we seen this expression in the context of series? Yes, recall that when |r| < 1, we have the following for geometric series

$$\sum_{n=0}^{\infty} ar^n = \sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}.$$

This implies that

$$f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n,$$

as long as |x| < 1. That is, f has a power series representation centered at 0 with radius 1. Outside the interval (-1, 1) the two expression are not equal.

Let's take a look at what is going on graphically by looking at the Power Series and Interval of Convergence Applet, which is available at http://calculusapplets.com/.

Not every power series takes the form of a geometric series. So, we need a more general method.

Taylor & Maclaurin Series

Suppose f has a power series representation (with |x - a| < R):

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \cdots$$

Since these functions are equal, their derivatives agree:

$$f'(x) = \sum_{n=1}^{\infty} nc_n (x-a)^{n-1} = c_1 + 2c_2 (x-a) + 3c_3 (x-a)^2 + \cdots$$

This implies that

$$f'(a) = c_1 + 0 + 0 + \dots = c_1.$$

That is,



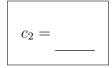
Let's repeat this process with the second derivative. We see that

$$f''(x) = \sum_{n=2}^{\infty} n(n-1)c_n(x-a)^{n-2} =$$

This implies that

$$f''(a) = \underline{\qquad} = \underline{\qquad}.$$

Then



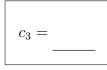
And again using the third derivative:

$$f'''(x) = \sum_{n=3}^{\infty} n(n-1)(n-2)c_n(x-a)^{n-3} =$$

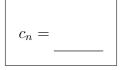
This implies that

$$f^{\prime\prime\prime}(a) = ____= ___$$

Then



If we continue this way, we'll obtain the following



Note 4. Here are a couple of conventions:

- 1. 0! = 1! = 1
- 2. $f^{(0)}(a) = f(a)$

Theorem 5. If f has a power series representation at x = a, then it must be of the form

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

for |x - a| < R, where R is the radius of convergence.

The above power series is called the Taylor Series of f centered at x = a. In the special case that a = 0, we get

$$f(x) = \underbrace{,}$$

which is called the Maclaurin series of f.

Important Note 6. We can always compute a Taylor/Maclaurin series, but that does not mean that it is equal to the given function. We only know that a Taylor/Maclaurin Series is equal to a given function *if* the given function can be represented by a power series. We will only deal with these types of functions.

Example 7.

(a) Find Maclaurin series for $f(x) = e^x$ and its radius of convergence (given that f has a power series representation).

(b) Find Maclaurin series for $f(x) = \sin x$ and its radius of convergence (given that f has a power series representation).

Common Taylor Series

Here are some common Taylor Series.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad \qquad R = 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \qquad R = \infty$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \qquad \qquad R = \infty$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \qquad \qquad R = \infty$$

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \qquad \qquad R = 1$$

Taylor Series approximation

If a function has a Taylor Series representation, then we can use a finite number of terms to approximate the function. We define the *kth degree Taylor polynomial of f at* x = a to be

$$T_k(x) = \sum_{n=0}^k \frac{f^{(n)}(a)}{n!} (x-a)^n$$

Let's take a look at the Taylor Series and Polynomials Applet available at http://calculusapplets.com/.

Example 8.

(a) Use the 9th degree Taylor polynomial for $\arctan x$ to approximate π .

(b) Approximate the following integral using a 5th degree Taylor polynomial for $\sin x$.

$$\int_0^1 x \sin(x^3) \ dx$$