Section 12.10: Taylor & Maclaurin Series

Goal

We will introduce Taylor and Maclaurin Series, which are special types of power series.

Initial discussion

Recall 1. A *power series* is an infinite power series of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

We say that the power series is centered at $x = a$.

Question 2. What functions have power series representations?

Example 3. Consider the function

$$f(x) = \frac{1}{1-x}.$$

Have we seen this expression in the context of series? Yes, recall that when $|r| < 1$, we have the following for geometric series

$$\sum_{n=0}^{\infty} ar^n = \sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}.$$

This implies that

$$f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n,$$

as long as $|x| < 1$. That is, f has a power series representation centered at 0 with radius 1. Outside the interval $(-1, 1)$ the two expression are not equal.

Let's take a look at what is going on graphically by looking at the Power Series and Interval of Convergence Applet, which is available at http://calculusapplets.com/.

Not every power series takes the form of a geometric series. So, we need a more general method.

Taylor & Maclaurin Series

Suppose f has a power series representation (with $|x-a| < R$):

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

Since these functions are equal, their derivatives agree:

$$f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1} = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots$$

This implies that

$$f'(a) = c_1 + 0 + 0 + \cdots = c_1.$$

That is,

$$\boxed{c_1 = \underline{\hspace{2em}}}$$

Let's repeat this process with the second derivative. We see that

$$f''(x) = \sum_{n=2}^{\infty} n(n-1)c_n(x-a)^{n-2} = \underline{\hspace{20em}}$$

This implies that

$$f''(a) = \underline{\hspace{8em}} = \underline{\hspace{6em}}.$$

Then

$$\boxed{c_2 = \frac{\hspace{2em}}{\hspace{2em}}}$$

And again using the third derivative:

$$f'''(x) = \sum_{n=3}^{\infty} n(n-1)(n-2)c_n(x-a)^{n-3} = \underline{\hspace{20em}}$$

This implies that

$$f'''(a) = \underline{\hspace{8em}} = \underline{\hspace{6em}}.$$

Then

$$\boxed{c_3 = \frac{\hspace{2em}}{\hspace{2em}}}$$

If we continue this way, we'll obtain the following

$$\boxed{c_n = \frac{\hspace{2em}}{\hspace{2em}}}$$

Note 4. Here are a couple of conventions:

1. $0! = 1! = 1$
2. $f^{(0)}(a) = f(a)$

Theorem 5. *If f has a power series representation at $x = a$, then it must be of the form*

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n}$$

for $|x - a| < R$, where R is the radius of convergence.

The above power series is called the *Taylor Series of f centered at $x = a$.* In the special case that $a = 0$, we get

$$\boxed{f(x) = \underline{\hspace{4cm}},}$$

which is called the *Maclaurin series of f.*

Important Note 6. We can always compute a Taylor/Maclaurin series, but that does not mean that it is equal to the given function. We only know that a Taylor/Maclaurin Series is equal to a given function *if* the given function can be represented by a power series. We will only deal with these types of functions.

Example 7.

(a) Find Maclaurin series for $f(x) = e^x$ and its radius of convergence (given that f has a power series representation).

(b) Find Maclaurin series for $f(x) = \sin x$ and its radius of convergence (given that f has a power series representation).

Common Taylor Series

Here are some common Taylor Series.

$\displaystyle \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$	$R = 1$
$\displaystyle e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$	$R = \infty$
$\displaystyle \sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$	$R = \infty$
$\displaystyle \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$	$R = \infty$
$\displaystyle \arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$	$R = 1$

Taylor Series approximation

If a function has a Taylor Series representation, then we can use a finite number of terms to approximate the function. We define the *kth degree Taylor polynomial of f at $x = a$* to be

$$\boxed{T_k(x) = \sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!}(x-a)^n}$$

Let's take a look at the Taylor Series and Polynomials Applet available at http://calculusapplets.com/.

Example 8.

(a) Use the 9th degree Taylor polynomial for $\arctan x$ to approximate π.

(b) Approximate the following integral using a 5th degree Taylor polynomial for $\sin x$.

$$\int_0^1 x\sin(x^3)\ dx$$