Cyclically Fully Commutative elements in Coxeter groups

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Basic Definitions

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A Coxeter group is a group W generaed by a finite set S of involutions with presentation

$$W = \left\langle S \mid s^2 = 1,\ (st)^{m(s,t)} = 1 \right\rangle,$$

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$$\left.\begin{array}{lcc} m(s,t) = 3 & \Longrightarrow & sts = tst \\ m(s,t) = 4 & \Longrightarrow & stst = tsts \\ & \vdots & \end{array}\right\} \quad \text{long braid relations}$$

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Motivation: Let's start by studying elements where long braid relations don't arise.

Cyclically reduced words

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If $u = s_{x_1} s_{x_2} \cdots s_{x_m} \in S^*$ and every cyclic shift of u is a reduced expression for some element in W, then u is cyclically reduced.

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(Non) Example

- $s_2 s_3 s_1 s_2 \in A_3$ is not cyclically reduced.

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- $s_1 s_3 s_1 s_2 \in \widetilde{A}_2$ is cyclically reduced, but not FC.
- $s_3 s_2 s_1 s_4 s_3 s_2 \in \widetilde{A}_3$ is FC, but not CFC.

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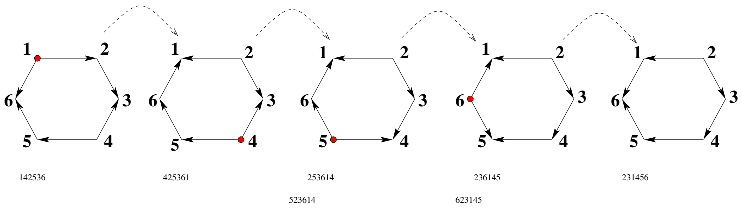
  Example. Let $W = \widetilde{A}_5$.



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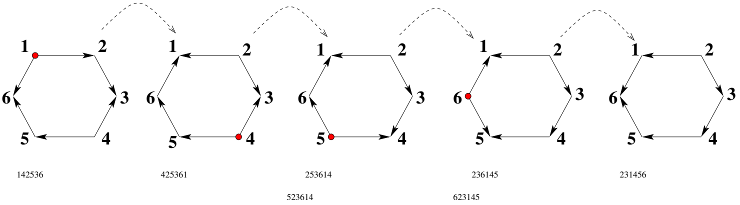

- Consider $w = ababcd$ in H_4. Because w is CFC, we can still play this "source-to-sink game."

CFC-finite groups

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Theorem

The irreducible CFC-finite Coxeter groups are:

A_n $(n \geq 1)$, B_n $(n \geq 2)$, D_n $(n \geq 4)$, E_n $(n \geq 6)$, F_n $(n \geq 4)$, H_n $(n \geq 3)$ *and* $I_2(m)$ $(5 \leq m < \infty)$.

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These are precisely the FC-finite Coxeter groups! (Stembridge, 1996).

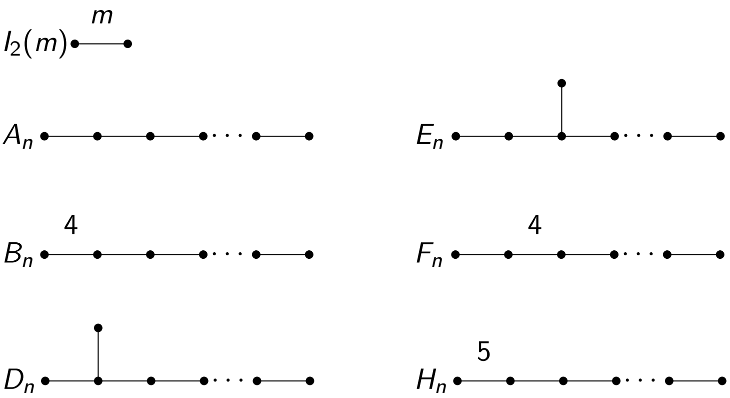

Figure: Connected Coxeter graphs corresponding to CFC-finite groups.

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In contrast, enumerating the FC elements (Stembridge, 1998) is very complicated.

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Can we count the CFC elements in $\widetilde{A}_{n-1}$ of a fixed length?

	$n=3$	4	5	6	7	n
$\ell(w)=1$	3	4	5	6	7	n
2	6	10	15	21	28	$\frac{n(n+1)}{2}$
3	6	16	30	50	77	
4	0	14	40	108	182	
5	0	0	30	96	336	
6	6	0	0	62	224	
7	0	0	0	0	126	
8	0	14	0	0	0	
9	6	0	0	0	0	
10	0	0	30	0	0	
11	0	0	0	0	0	
12	6	14	0	62	0	
n						2^n-2

CFC elements in $\widetilde{A}_{n-1}$

Remark. There are $2^n - 2$ Coxeter elements in $\widetilde{A}_{n-1}$.

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If $w \in CFC(\widetilde{A}_{n-1})$ and $\ell(w) \geq n$, then $w = c^k$ for some Coxeter element $c \in \mathrm{C}(\widetilde{A}_{n-1})$.

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Question: Are the any other Coxeter groups that have only finitely many *primitive* CFC elements, i.e.,

$$\lim_{k \to \infty} \frac{|w \in \mathrm{CFC}(W) \mid \ell(w) \leq k|}{k} = \mathcal{O}(1)?$$

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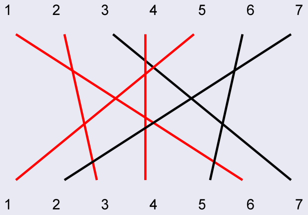

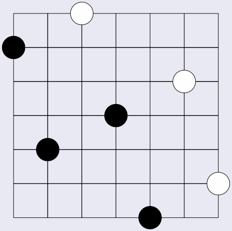

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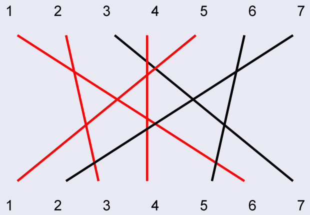

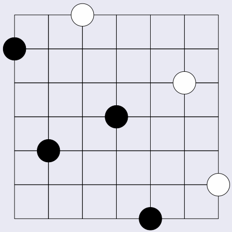

- σ contains 4231, avoids 4321.

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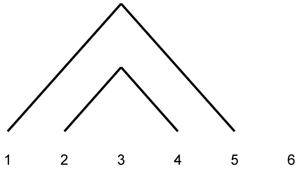
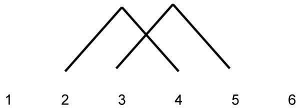
Question (Billey): Can we characterize the CFC property using "generalized pattern avoidance" in other Coxeter groups?

Type A_n and the Catalan numbers

In type A_n, the following quantities are all counted by the Catalan numbers:

- The fully commutative elements
- The non-crossing partitions
- The non-nesting partitions.





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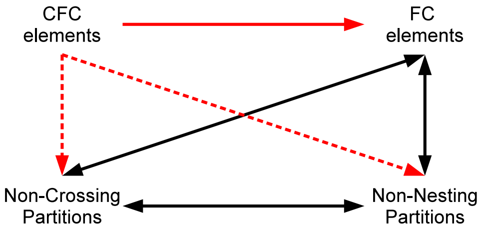
▶ What do the "CFC" non-nesting and non-crossing partitions look like? If this is interesting, can we extend it beyond type A_n?



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