Classification of the T-avoiding permutations and generalizations to other Coxeter groups

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Definition

The symmetric group S_n is the collection of bijections from $\{1, 2, \ldots, n\}$ to $\{1, 2, \ldots, n\}$ where the operation is function composition (left $\leftarrow$ right). Each element of S_n is called a permutation.

Comment

We can think of S_n as the group that acts by rearranging n coins.

One way of representing permutations is via cycle notation, which we will illustrate by way of example.

Example

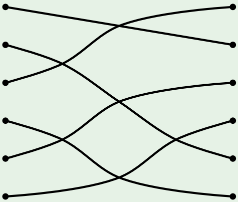
Consider $\sigma = (1\ 3\ 5\ 2)(4\ 6)$. This means $\sigma(1) = 3$, $\sigma(3) = 5$, $\sigma(5) = 2$, $\sigma(2) = 1$, $\sigma(4) = 6$, and $\sigma(6) = 4$.

String diagrams

A second way of representing permutations is via string diagrams, which we again introduce by way of example.

Example

Consider $\sigma = (1\ 3\ 5\ 2)(4\ 6)$ from previous example.



Comment

Given a permutation σ, there are many ways to draw the associated string diagram. However, we adopt the following conventions:

1. no more than two strings cross each other at a given point,
2. strings are drawn so as to minimize crossings.

S_n is generated by the adjacent 2-cycles:

$$(1\ 2), (2\ 3), \ldots, (n-1\ n).$$

That is, every element of S_n can be written as a product of the adjacent 2-cycles.

Define

$$s_i = (i\ i+1)$$

so that $s_1, s_2, \ldots, s_{n-1}$ generate S_n.

Comments

S_n satisfies the following relations:

1. $s_i^2 = 1$ for all i (2-cycles have order 2)
2. short braid relations: $s_i s_j = s_j s_i$, for $|i-j| \geq 2$ (disjoint cycles commute)
3. long braid relation: $s_i s_j s_i = s_j s_i s_j$, for $|i-j| = 1$.

Definition

If $s_{x_1} s_{x_2} \cdots s_{x_m}$ is an expression for $\sigma \in S_n$ and m is minimal, then we say that the expression is reduced.

Example

Consider $\sigma = s_2 s_1 s_2 s_3 s_1 s_2 \in S_4$. We see that

$$s_2 s_1 s_2 s_3 s_1 s_2 = s_1 s_2 s_1 s_3 s_1 s_2 = s_1 s_2 s_1 s_3 s_1 s_2 = s_1 s_2 s_1 s_1 s_3 s_2 = s_1 s_2 s_1 s_1 s_3 s_2 = s_1 s_2 s_3 s_2.$$

So, the original expression was *not* reduced, but it turns out that the last expression on the right is reduced.

Theorem (Matsumoto)

Any two reduced expressions for $\sigma \in S_n$ differ by a sequence of braid relations.

Example (continued)

The only reduced expressions for σ are: $s_1 s_2 s_3 s_2$, $s_1 s_3 s_2 s_3$, and $s_3 s_1 s_2 s_3$.

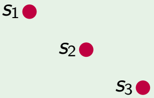
Heaps

A third way of representing permutations is via heaps. Fix a reduced expression $s_{x_1} s_{x_2} \cdots s_{x_m}$ for $\sigma \in S_n$. Loosely speaking, the heap for this expression is a set of lattice points (called nodes), one for each s_{x_i}, embedded in $\mathbb{N} \times \mathbb{N}$ such that:

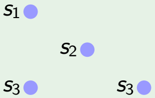
- The node corresponding to s_{x_i} has vertical component equal to $n + 1 - x_i$ (smaller numbers at the top),
- If $i < j$ and s_{x_i} does not commute with s_{x_j}, then s_{x_i} occurs to the left of s_{x_j}.

Example

Consider $s_1 s_2 s_3 s_2$, $s_1 s_3 s_2 s_3$, and $s_3 s_1 s_2 s_3$ from the previous example. It turns out, there are two distinct heaps.



and



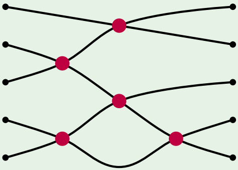
Comment

If two reduced expressions for σ differ by a sequence of short braid relations, then they have the same heap. In particular, if no reduced expression contains an opportunity to provide a long braid relation, then σ has a unique heap.

The points at which two strings cross correspond to nodes in the heap. Hence, we may overlay strings on top of a heap by drawing the strings from right to left so that they cross at each entry in the heap where they meet and bounce at each lattice point not in the heap.

Conversely, each string diagram corresponds to a heap by taking all of the points where the strings cross as the nodes of the heap.

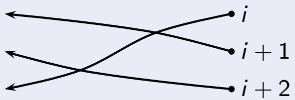
Example



Definition

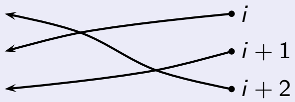
We say that a permutation σ has Property T iff there exists i such that either

1. $\sigma(i) > \sigma(i+1), \sigma(i+2)$,



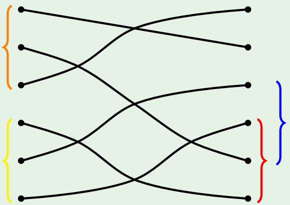
or

2. $\sigma(i+2) < \sigma(i), \sigma(i+1)$.



Example

Consider the following permutation $\sigma = (1\ 3\ 5\ 2)(4\ 6)$.



We see that σ and σ^{-1} each have Property T in two spots.

T-avoiding

Definition

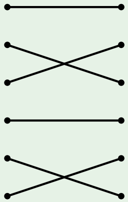
We say that a permutation σ is T-avoiding iff neither σ or σ^{-1} has Property T.

Theorem (CEGKM)

A permutation σ is T-avoiding iff σ is a product of disjoint adjacent 2-cycles.

Example

The permutation $\sigma = (2\ 3)(5\ 6)$ is T-avoiding.



Sketch of proof

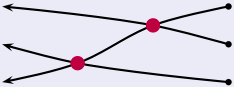
Fix a reduced expression for σ, say $s_{x_1} s_{x_2} \cdots s_{x_m}$, and consider the heap for this reduced expression. The reverse implication of the theorem is trivial. For the forward direction, consider the contrapositive:

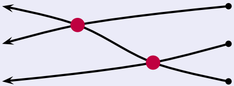
If σ is not a product of disjoint adjacent 2-cycles, then σ or σ^{-1} has Property T.

The easy case

The easy case occurs when a node in the second column on either side is "blocked" by at most one node in the first column.

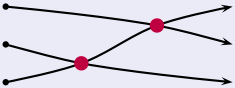
1. σ has the property:

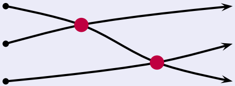




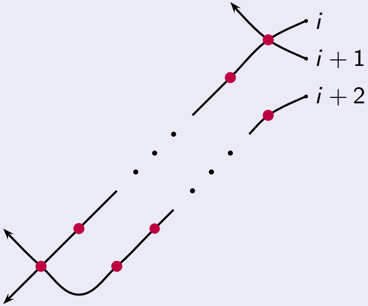
or

2. σ^{-1} has the property:





The hard case



Theorem

By applying a sequence of long braid relations, you can convert a heap in the hard case to a heap in the easy case.

The previous theorem provides the necessary motivation for generalizing the definition of Property T in other Coxeter groups.

Definition

A Coxeter group consists of a group W together with a generating set S consisting of elements of order 2 with presentation

$$W = \langle S : s^2 = 1, (st)^{m(s,t)} = 1 \rangle,$$

where $m(s,t) \geq 2$ for $s \neq t$.

Comment

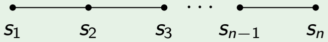
Since s and t are elements of order 2, the relation $(st)^{m(s,t)} = 1$ can be rewritten as

$$\begin{aligned} m(s,t) = 2 &\implies st = ts \quad \} \text{ short braid relations} \\ m(s,t) = 3 &\implies sts = tst \\ m(s,t) = 4 &\implies stst = tsts \\ &\vdots \end{aligned} \left.\vphantom{\begin{matrix} a \\ a \\ a \end{matrix}}\right\} \text{ long braid relations}$$

We can uniquely encode the generators and relations using a Coxeter graph.

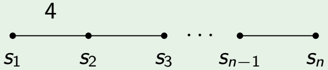
Example (Type A)

The symmetric group S_{n+1} with the adjacent 2-cycles as a generating set is a Coxeter group of type A_n.



Example (Type B)

Coxeter groups of type B_n ($n \geq 2$) are defined by:



$W(B_n)$ is generated by $S(B_n) = \{s_1, s_2, \cdots, s_n\}$ and is subject to

1. $s_i^2 = 1$ for all i,
2. $s_i s_j = s_j s_i$ if $|i - j| > 1$,
3. $s_i s_j s_i = s_j s_i s_j$ if $|i - j| = 1$ and $1 < i, j \leq n$,
4. $s_1 s_2 s_1 s_2 = s_2 s_1 s_2 s_1$.

Definition

Let (W, S) be a Coxeter group and let $w \in W$. Then w has Property T iff w has a reduced expression of the form stu or uts, where $m(s,t) \geq 3$ and $u \in W$.

Theorem (CEGKM)

In type A and B, $w \in W$ is T-avoiding iff w is a product of commuting generators.

Comment

The answer isn't so simple in other Coxeter groups.

- We have also classified the T-avoiding elements in type affine C, which consists of more than just products of commuting generators.
- Similarly, Tyson Gern has recently classified the T-avoiding elements in type D, and again, the classification is more complicated than just products of commuting generators.