Pattern Avoidance and Affine Permutations

Joint work with Sara Billey

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April, 10 2011

Definition of Affine Permutations

Definition

The group of affine permutations, $\widetilde{S}_n$ is the group of all bijections $\sigma : \mathbb{Z} \to \mathbb{Z}$ such that the following properties hold:

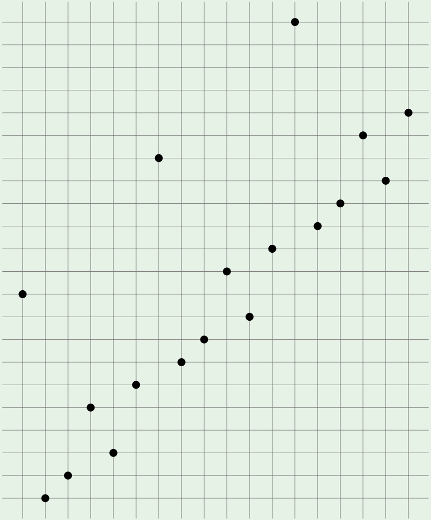
1. $\sigma(i+n) = \sigma(i) + n$ for all $i \in \mathbb{Z}$,
2. $\sum_{i=1}^{n} \sigma(i) = \binom{n+1}{2}$.

We will represent an affine permutation in its one-line notation as the infinite string

$$\cdots \sigma(-1), \sigma(0) \, [\sigma(1), \ldots, \sigma(n)] \, \sigma(n+1), \sigma(n+2) \cdots .$$

Example

$$\sigma = \ldots, 2, -7, -6, -3, -5, -2[8, -1, 0, 3, 1, 4]14, 5, 6, 9, 7, 10, \ldots \in \widetilde{S}_6.$$



Coxeter Groups

As a Coxeter group, $\widetilde{S}_n$ is generated by the simple reflections $S = \{s_0, s_1, \ldots, s_{n-1}\}$, where s_i interchanges $i + kn$ and $i + 1 + kn$ for all $k \in \mathbb{Z}$ and leaves all other integers fixed. The relations amongst these generators is summarized in the Coxeter graph.

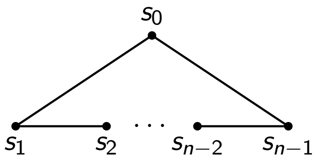


Figure: Coxeter graph for $\widetilde{S}_n$.

Definition of Pattern Avoidance

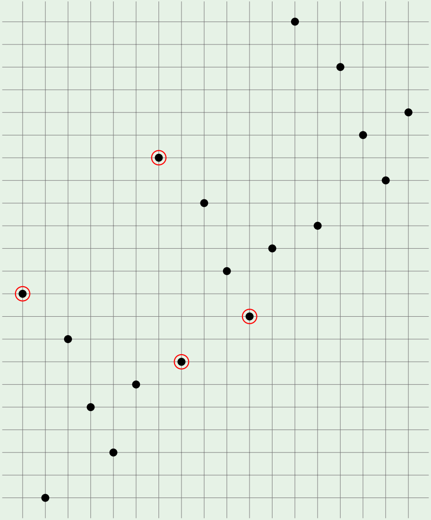
Mimicking the notion of pattern avoidance for non-affine permutations, we make the following definition.

Definition

Let $\sigma \in \widetilde{S}_n$ and $p \in S_m$. We say σ *contains* the pattern p if there exist indices $i_1 < \cdots < i_m$, such that the values $\sigma(i_1), \ldots, \sigma(i_m)$ are in the same relative order as the values $p(1), \ldots, p(m)$. Otherwise, we say σ *avoids* the pattern p.

Example

$\sigma = [8, -1, 6, 3, 1, 4]$ contains $p = 3412$.



Spiral Permutations

Definition

Pick any $1 \leq i \leq n$. Starting at s_i, proceed clockwise or counterclockwise $k(n-1)$ steps, building a word from right to left at each step. The resulting affine permutation is called a *spiral permutation*.

Example

$$\sigma = s_2 s_1 s_0 s_3 s_2 s_1 s_0 s_3 s_2 = [-2, 11, 0, 1]2, 15, 4, 5, \ldots \in \widetilde{S}_4.$$

Twisted Spiral Permutations

Definition

A *twisted spiral permutation* is obtained by taking a spiral permutation starting at s_i, and multiplying on the right by the long element of the maximal parabolic subgroup generated by $S \backslash \{s_i\}$.

Example

$$\sigma \cdot s_3 s_0 s_3 s_1 s_0 s_3 = [-3, -4, 15, 2].$$

Enumeration of Pattern Avoidance

Let

$$\widetilde{S}_n(p) = \left\{\sigma \in \widetilde{S}_n : \sigma \text{ avoids } p\right\}.$$

Since $\widetilde{S}_n$ is an infinite group, for a given pattern $p \in S_m$, $\widetilde{S}_n(p)$ could be infinite.

Theorem (Crites 2010)

$\widetilde{S}_n(p)$ is finite if and only if p avoids the pattern 321.

For example,

$$\widetilde{S}_n(231) = \widetilde{S}_n(312) = \binom{2n-1}{n}.$$

Affine Permutation Matrices

Write $\sigma(i) = a_i + b_i n$, where $1 \leq a_i \leq n$. Since σ is a bijection, Property 1 guarantees that $\{a_1, \ldots, a_n\} = \{1, \ldots, n\}$. Property 2 shows that $b_1 + \cdots + b_n = 0$.

Definition

Let $e_\sigma = (m_{ij})_{i,j=1}^n$ be the matrix with $m_{i,a_i} = t^{b_i}$ for $1 \leq i \leq n$, and all other entries 0. Such a matrix is called an *affine permutation matrix*.

Example

$$\sigma = [\ 8,\ -1,\ 6,\ 3,\ 1,\ 4\ ]$$
$$a_i:\ 2\quad 5\quad 6\quad 3\quad 1\quad 4$$
$$b_i:\ 1\quad -1\quad 0\quad 0\quad 0\quad 0$$
$$\updownarrow$$
$$\begin{bmatrix} 0 & t & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & t^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Affine Schubert Varieties

Let

$$\widetilde{G} = \mathrm{GL}_n\left(\mathbb{C}[[t]][t^{-1}]\right),$$

and let

$$\widetilde{B} = \{b \in \mathrm{GL}_n(\mathbb{C}[[t]]) : \left.b\right|_{t=0} \text{ is upper triangular}\}.$$

Finally, let

$$\widetilde{G}_0 = \left\{g \in \widetilde{G} : \operatorname{ord}\det g = 0\right\}.$$

Definition

$\widetilde{X} := \widetilde{G}_0/\widetilde{B}$ is an ind-variety called the *complete affine flag variety*. The corresponding affine Weyl group is $\widetilde{S}_n$.

Affine Schubert Varieties (cont.)

By putting elements of $\widetilde{G}_0$ in column echelon form, we have the *Bruhat decomposition*

$$\widetilde{X} = \bigsqcup_{\sigma \in \widetilde{S}_n} \widetilde{B}\sigma\widetilde{B}/\widetilde{B}.$$

Definition

1. The *Schubert cell* corresponding to $\sigma \in \widetilde{S}_n$ is $C_\sigma = \widetilde{B}\sigma\widetilde{B}/\widetilde{B}$.
2. The *Schubert variety* corresponding to σ is $\widetilde{X}_\sigma = \overline{C}_\sigma$.

Example

The Schubert cell C_σ, for $\sigma = [8, -1, 6, 3, 1, 4]$:

$$\begin{bmatrix} a+bt & t & c & d & et^{-1}+f & g \\ 0 & 0 & 0 & 0 & t^{-1} & 0 \\ h & 0 & i & j & k & 1 \\ \ell & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

The number of free variables in row i is

$$\#\left\{j : i < j \text{ and } \sigma(i) > \sigma(j)\right\}.$$

Affine Schubert Varieties (cont.)

As in the classical case, we have the following properties:

$$\widetilde{X}_\tau = \bigcup_{\sigma \leq \tau} \widetilde{X}_\sigma,$$

$$\dim \widetilde{X}_\sigma = \ell(\sigma) = \# \left\{ (i,j) : 1 \leq i \leq n, i < j, \sigma(i) > \sigma(j) \right\}.$$

Rational Smoothness

Definition

The *Poincaré polynomial* for $\tau \in \widetilde{S}_n$ is given by

$$P_\tau(q) = \sum_{\sigma \leq \tau} q^{\ell(\sigma)}.$$

Definition

A variety X is *rationally smooth* if, for each $x \in X$, the singular cohomology $H^i(X, X \setminus \{x\}, \mathbb{Q}) = 0$ for $i \neq 2 \dim X$, and is one-dimensional when $i = 2 \dim X$.

Theorem (Carrell-Peterson)

$\widetilde{X}_\sigma$ is rationally smooth if and only if $P_\sigma(q)$ is palindromic.

Main Result

Theorem (Billey-Crites 2010)

The affine Schubert variety $\widetilde{X}_\sigma$ is rationally smooth if and only if either σ avoids 3412 and 4231, or σ is a twisted spiral variety.

Since smoothness implies rational smoothness, we also get the following.

Corollary

If σ contains 3412 or 4231, then $\widetilde{X}_\sigma$ is not smooth.

Proof Idea

- If σ avoids 3412 and 4231, we can extend Gasharov's proof from the classical case to give a factorization.
- Rational smoothness of spiral varieties follows from work of Billey-Mitchell in the affine Grassmannian. These varieties are then lifted to the complete affine flag variety via twisting.
- If σ contains 4231, then the Poincaré polynomial fails to be palindromic at q and $q^{\ell(\sigma)-1}$, so if suffices to count the number of elements that σ covers.
- If σ contains 3412, we extend the results of Billey-Warrington on Bruhat pictures to find specific rationally singular points.

Factorization Formula

Let $\sigma \in \widetilde{S}_n$. For $1 \leq i \leq n$, let $r \leq i < r'$ be left-to-right maxima such that no other left-to-right maxima lies between r and r'. Let

$$e_i = \#\{r \leq j < i : \sigma(j) > \sigma(i)\} + \#\left\{r' < j : \sigma(i) > \sigma(j)\right\} + 1.$$

Let

$$[m]_q = 1 + q + \cdots + q^{m-1}$$

be the q-analog of m.

Corollary

If $\sigma \in \widetilde{S}_n$ avoids 3412 and 4231, then

$$P_\sigma(q) = \prod_{i=1}^{n} [e_i]_q.$$

Example

Let σ be the affine permutation

$$-8, -7, \underline{-2}, -3, -4, \underline{2}, -5, -1[0, 1, \underline{6}, 5, 4, \underline{10}, 3, 7]8, 9, \underline{14}, 13, 12, \underline{18}, 11, 15$$

Note that σ avoids 3412 and 4231. The values of e_i are $(2, 2, 2, 3, 4, 0, 2, 2)$, so the Poincaré polynomial factors as

$$P_\sigma(q) = (1+q)^5(1+q+q^2)(1+q+q^2+q^3).$$

Comparison with Classical Case

For non-affine permutations, smoothness and rational smoothness are equivalent.

Theorem (Lakshmibai-Sandhya)

Let $\sigma \in S_n$. Then X_σ is (rationally) smooth if and only if σ avoids 3412 and 4231.

This might lead us to make the following conjecture.

Conjecture

Let $\sigma \in \widetilde{S}_n$. Then $\widetilde{X}_\sigma$ is smooth if and only if σ avoids 3412 and 4231.

Smoothness

Theorem (Cheng-Crites-Kuttler)

Let $\sigma \in \widetilde{S}_n$. $\widetilde{X}_\sigma$ is smooth if and only if σ avoids 3412 and 4231. In other words, the twisted spiral varieties are the only ones that are rationally smooth, but not smooth.

Smoothness is much harder to detect from the combinatorics of $\widetilde{S}_n$ due to the existence of *imaginary roots*. These roots contribute to the dimension of the tangent space, but do not correspond to reflections in the Weyl group.

Thank you