A refinement of weak order intervals into distributive lattices

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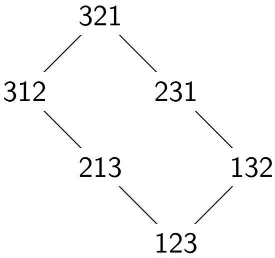


Let s_i denote the adjacent transposition $(i \;\; i+1)$.

The length $\ell(w)$ of $w \in S_n$ is the minimum number of adjacent transpositions required to express an element.

Left weak Bruhat order is defined by the covering relations

$$v \prec w \iff w = s_i v \text{ and } \ell(w) = \ell(v) + 1.$$

So $s_1 < s_2 s_1$ ($213 < 312$) in the left weak Bruhat order.

An inversion of $w \in S_n$ is a pair (i, j) such that $i < j$ and $w(i) > w(j)$.

Example

(1,3) is an inversion of 2413 $\rightsquigarrow$ 2413
(2,3) is an inversion of 2413 $\rightsquigarrow$ 2413
(2,4) is an inversion of 2413 $\rightsquigarrow$ 2413

The inversion set of $w \in S_n$ is the set of all inversions of w.

Well Known Fact

A permutation is uniquely determined by its inversion set.

The Lehmer code of $w \in S_n$ is the n-tuple $(c_1, \ldots, c_n)$, where c_i is the number of inversions whose first coordinate is i.

Example

From the previous slide, the inversion set of 2413 is

$$\{(1,3),(2,3),(2,4)\}.$$

The Lehmer code of 2413 is (1,2,0,0).

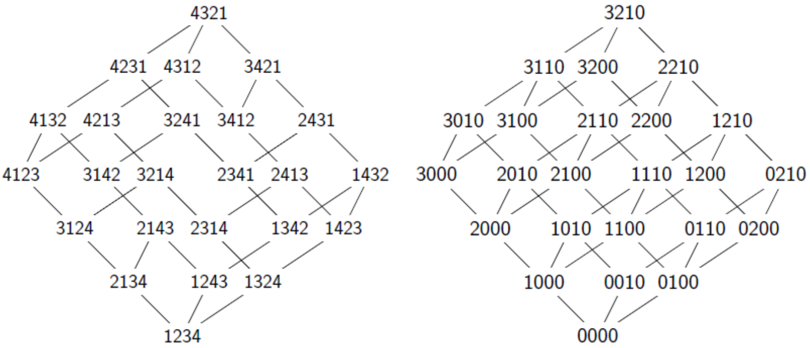
The Lehmer code bijection

1. The Lehmer code determines a bijection

$$\mathbf{c} : S_n \to \prod_{i=1}^{n}[0, n-i]$$

2. If $v < w$ in the weak order, then $\mathbf{c}(v) < \mathbf{c}(w)$ in the product order. The converse is false.

The weak order on S_4 versus the product order on $[0,3] \times [0,2] \times [0,1]$.



The rank-generating function of the weak order on S_4 is given by

$$F(q) = (1+q)(1+q+q^2)(1+q+q^2+q^3).$$

Some open questions regarding the interval Λ_w and its rank-generating function

These are from a recent paper of Wei's:

1. For what $w \in S_n$ is the rank-generating function $F(\Lambda_w, q)$ rank-symmetric (i.e. palindromic)?
2. When is $(1+q)(1+q+q^2)(\ldots)$ divisible by $F(\Lambda_w, q)$?
3. When is $F(\Lambda_w, q)$ a product of cyclotomic polynomials?

Other interesting questions:

1. For which permutations $u, v \in S_n$ do we have $F(\Lambda_u, q) = F(\Lambda_v, q)$?
2. Given $w \in S_n$, what is the minimum dimension d such that Λ_w can be embedded in $\mathbb{N}^d$.

The FTFDL (Birkhoff)

A distributive lattice satisfies the distributive laws:

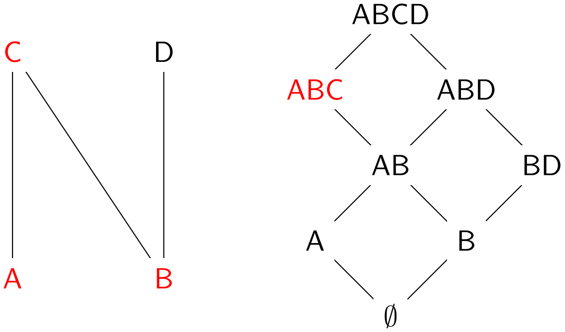
$$x \vee (y \wedge z) = (x \vee y) \wedge (x \wedge z)$$
$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

Every distributive lattice can be realized as a lattice of sets, where $\vee \leftrightarrow \cup$ and $\wedge \leftrightarrow \cap$ and $\leq \leftrightarrow \subseteq$. What is this lattice of sets like?

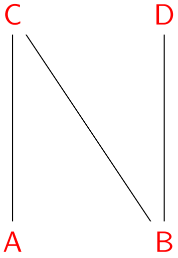
For every finite distributive lattice L, there is a poset P such that $L \cong J(P)$, where $J(P)$ is the set of all order ideals of P.

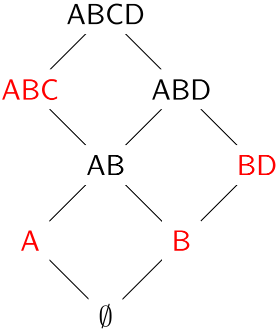
order ideal	$\leftrightarrow$	element/vertex
element/vertex	$\leftrightarrow$	join-irreducible
natural labeling	$\leftrightarrow$	maximal chain
order-preserving map	$\leftrightarrow$	multichain
antichain	$\leftrightarrow$	boolean sublattice

order ideal $\leftrightarrow$ element/vertex

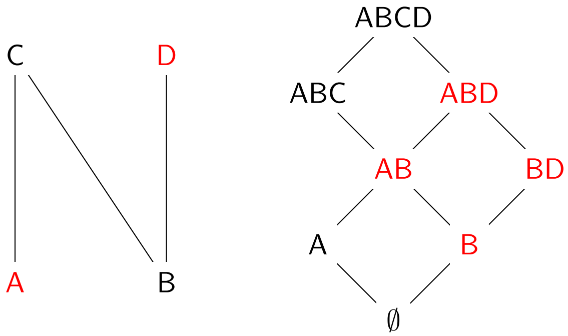


elements/vertices $\leftrightarrow$ join-irreducibles





antichain ↔ boolean sublattice



The extended Lehmer code

Let $w \in S_n$. We denote the i-th coordinate of the Lehmer code by $c_i(w)$.

Let $c_{i,j}(w)$ be the number of inversions whose first coordinate is i and whose second coordinate is less than j.

We call the matrix of $c_{i,j}(w)$'s the extended Lehmer code of w.

Example

Let $w = 561324$. Then $c_1(w) = 4$ and $c_{1,5}(w) = 2$.

561324

Detecting inversions with the codes

If $i < j$ then $c_i(w) \leq c_j(w) + c_{i,j}(w) \iff (i,j) \notin I(w)$.

A theorem

We denote the weak order interval $[\mathrm{id}, w]$ by Λ_w. Stembridge showed that Λ_w is distributive if and only if w is a fully commutative element.

We can use the extended Lehmer code to detect relations in the weak order. These statements are equivalent:

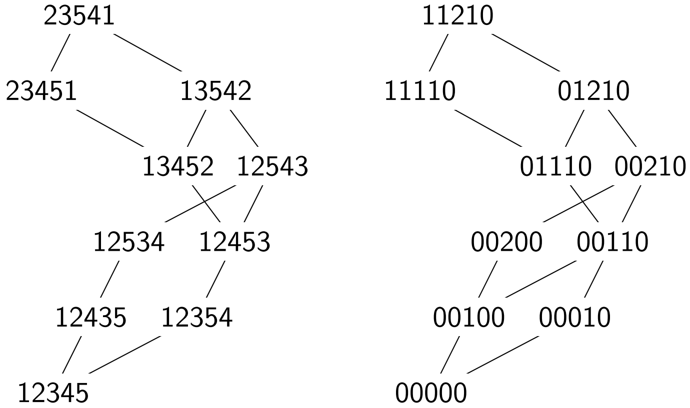
1. For every $(i,j) \notin I(w)$ we have $c_i(v) \leq c_j(v) + c_{i,j}(w)$.
2. In the weak Bruhat order, we have $v \leq w$.

From this technical fact, we can prove that the set of Lehmer codes are a sublattice of $\mathbb{N}^n$.

Theorem (D)

Let $w \in S_n$. The subposet $\mathbf{c}(\Lambda_w)$ of $\mathbb{N}^n$ is a distributive lattice.

Example



The interval [12345, 23541] in the weak order is not distributive.
The set **c**([12345, 23541]) of Lehmer codes is a distributive lattice.

For a given $w \in S_n$, can we describe the base poset?

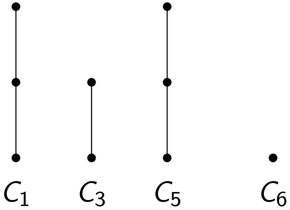
Let $L_w = \mathbf{c}(\Lambda_w)$. We let P_w denote a finite poset such that $L_w \cong J(P_w)$. What does P_w look like?

The P_w recipe

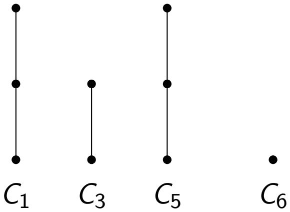
1. For each nonzero coordinate i of the Lehmer code, construct a chain $C_i(w)$ whose size is given by the code.
2. The x-th element of $C_i(w)$ and the y-th element of $C_j(w)$ are related if and only if

$$y \leq x - c_{i,j}(w).$$

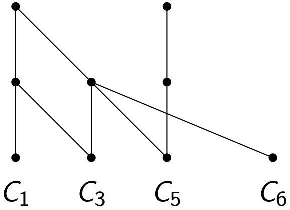
Let $w = 41528637$ so that $\mathbf{c}(w) = (3, 0, 2, 0, 3, 1, 0, 0)$. First we draw the chains.



Let $w = 41528637$ so that $\mathbf{c}(w) = (3, 0, 2, 0, 3, 1, 0, 0)$.



The non-inversions are $(1, 3)$, $(1, 5)$, $(1, 6)$, $(3, 5)$, and $(3, 6)$. The relevant extended codes are $c_{1,3}(w) = 1$, $c_{1,5}(w) = 2$, $c_{1,6}(w) = 2$, $c_{3,5}(w) = 1$, and $c_{3,6}(w) = 1$.



$$y \leq x - c_{i,j}(w)$$

Further Exploration I

A weak order interval is rank-symmetric if the poset P_w is self-dual. There are other known sufficient conditions (w is separable - Wei), but no known characterization.

Combining results due to Lakshmibai-Sandhya and Carrell-Peterson, an interval $[\mathrm{id}, w]$ in the *strong* Bruhat order is rank-symmetric if and only if w avoids 4231 and 3412.

There can be no pattern avoidance criteria for rank-symmetry of weak order intervals. For $u \in S_p$ and $v \in S_q$, let $u \times v$ denote the image of the usual embedding of $S_p \times S_q$ into S_{p+q}. The rank-generating functions $F(\Lambda_u, q)$ and $F(\Lambda_{u^{-1}}, q)$ are the reverse of one another. Therefore $F(\Lambda_{u \times u^{-1}}, q) = F(\Lambda_u, q)F(\Lambda_{u^{-1}}, q)$ is symmetric and $u \times u^{-1}$ contains u as a pattern.

Further Exploration II

In type D_4, under the usual labeling of the Coxeter graph, the element $s_2 s_1 s_3 s_4 s_2 s_4 s_3 s_1 s_2$ has a rank-generating function that is not the rank-generating function of a distributive lattice. This element arose in a paper of Green-Losonczy to demonstrate the existence of inversion triples that are not contractible. Do "contractible elements" always have the rank-generating function of a distributive lattice?

What about type B?

Can the results be obtained or rephrased using the ω-sorting orders of Armstrong?

Can the construction of this talk be used to calculate the order dimension of a weak order interval?