Zero-Hecke Monoids and Pattern Avoidance

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Zero-Hecke Monoids

Given W a Coxeter group with index set I, we can form the **Zero-Hecke Monoid** $H = H_0(W)$ generated by π_i for $i \in I$.

Relations are the same as in W, except that $\pi_i^2 = \pi_i$.

For $W = S_n$, H can be realized as the monoid of anti-sorting operators.

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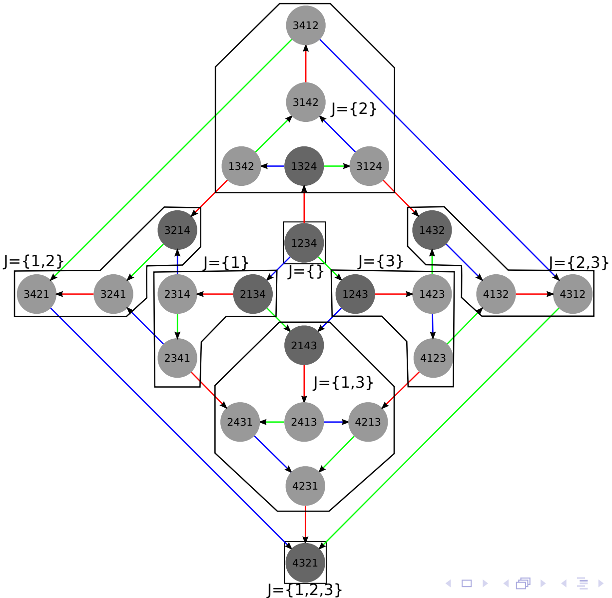
But this affords certain advantages... Strange things become possible!

Cayley Graph of H for $W = S_4$



The Parabolic Map

Question: Given $\sigma \in W$ and $J \subset I$, what is the longest element $\sigma_J \in W_J$ less than σ in Bruhat order?

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This is actually monoid morphism $H \to H_J$, the parabolic submonoid of H with generators indexed by J.
By comparison, a similar evaluation would fail miserably in the original Coxeter group.

Non-Decreasing Parking Functions

Define the **Non-Decreasing Parking Functions** (of type A_{n-1}) as the order-preserving functions from $[n] \to [n]$.

Generators: $\{f_i \mid i \in I\}$, defined by:

$$f_i(j) = \begin{cases} i & : j = i+1 \\ j & : j \neq i+1 \end{cases}$$

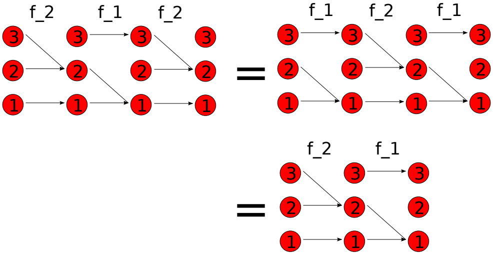
NDPF Relations

Definition

The **Non-Decreasing Parking Functions** (of type A_{n-1}) are functions $[n] \to [n]$ satisfying:

- $f(i) \leq i$
- $i \leq j \Rightarrow f(i) \leq f(j)$

Relations:



Fibers of the NDPF Quotient

In the NDPF quotient, elements π_w containing a braid occupy the same fiber as an element with that braid broken.

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Theorem (D.)

Every fiber of the NDPF quotient contains a unique 321-avoiding permutation of minimal length, and a unique 231-avoiding permutation of maximal length.

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Let F be a non-empty fiber of the map:

$$H_0(A_n) \to H_0(B_{n+1}) \to BNDPF_{n+1}.$$

Then F contains a unique 4321-avoiding permutation.

Affine NDPF

Definition

The **Extended Affine NDPF** (of type A_n) are the functions $f : \mathbb{Z} \to \mathbb{Z}$ satisfying:

1. $f(i) \leq i$
2. $i \leq j \Rightarrow f(i) \leq j(j)$
3. $f(i+n) = f(i) + n$
4. $f(n) - f(1) \leq n$

To get non-Extended version, throw away the ‘shift’ operators.

Combinatorial Quotient

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The upshot is the statement that each fiber of the affine NDPF quotient contains a unique 321-avoiding affine permutation!

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Thanks!