Polytopal subcomplexes and homology representations of Coxeter groups

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A Coxeter system is a pair (W, S), where W is a group given by the presentation

$$\langle S \mid (s_i s_j)^{m_{ij}} = 1 \rangle,$$

where $m_{ij} \in \mathbb{Z} \cup \{\infty\}$ and we have $m_{ii} = 1$ and $m_{ij} = m_{ji}$.
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Examples of Coxeter groups

The symmetric group, $W = \text{Sym}_{n+1}$, is an example of a Coxeter group, if we take $S = \{s_i : 1 \leq i \leq n\}$ where $s_i = (i, i+1)$.

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The group W is the automorphism group of the regular m-gon in 2 dimensions.

Examples of Coxeter groups (continued)

The hyperoctahedral group, $W = W(B_n)$, is the Coxeter group with generating set $S = \{s_1, s_2, \ldots, s_n\}$, where

$$m_{ij} = \begin{cases} 2 & \text{if } |i - j| > 1, \\ 4 & \text{if } \{i, j\} = \{1, 2\}, \\ 3 & \text{otherwise.} \end{cases}$$

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If $i > 1$, we may identify the generator s_i with the transposition $(i-1, i)$. The generator s_1 acts by sign change on the leftmost coordinate.

Examples of Coxeter groups (continued)

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A Coxeter group is finite if and only if the connected components of its Coxeter graph are finite in number and appear in a well known list (types A_n, B_n, D_n, E_6, E_7, E_8, F_4, H_3, H_4 and $I_2(m)$).

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A fundamental result in the theory of Coxeter groups is the following.

Theorem

Every finite reflection group is a finite Coxeter group, and vice versa. Furthermore, the Coxeter presentation of the group and the set of hyperplanes of the reflection group determine each other "up to isomorphism".

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For example, note that the vector $\mathbf{v} = (+2, +2, \dots, +2)$ is fixed by all the generators s_i except for $s_{1'}$, which moves it to $(-2, -2, +2, \dots, +2)$.

The reflection action of a finite Coxeter group (W, S) on $\mathbb{R}^n$ can be used to define a polytope by taking a vector $\mathbf{v} \in \mathbb{R}^n$ and then taking the convex hull, $\Pi(\mathbf{v})$, of the finite set of points $W.\mathbf{v}$.

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In order to appreciate what is going on, it is helpful to review the basics of the theory of polytopes.

Polytopes

If V is a vector space over $\mathbb{R}$ and $X \subset V$, then the affine hull, $\mathrm{Aff}(X)$, of X in V is the set of affine combinations of finite subsets of points in X; that is, the set of all vectors

$$\left\{ \sum_{i=1}^{k} \lambda_i x_i : x_i \in X,\ \lambda_i \in \mathbb{R},\ \sum_{i=1}^{k} \lambda_i = 1,\ k \in \mathbb{N} \right\}.$$

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The dimension of the affine hull of X is the dimension of the vector space of all differences of vectors in X; that is, the dimension of the space

$$\left\{ \sum_{i=1}^{k} \lambda_i x_i : x_i \in X,\ \lambda_i \in \mathbb{R},\ \sum_{i=1}^{k} \lambda_i = 0,\ k \in \mathbb{N} \right\}.$$

The convex hull, $\mathrm{Conv}(X)$, of X in V is the set of convex combinations of finite subsets of points in X; that is, the subset of $\mathrm{Aff}(X)$ given by the set

$$\left\{ \sum_{i=1}^{k} \lambda_i x_i : x_i \in X,\ \lambda_i \in \mathbb{R},\ \lambda_i \geq 0,\ \sum_{i=1}^{k} \lambda_i = 1,\ k \in \mathbb{N} \right\}.$$

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The automorphism group of a polytope is the subgroup of isometries of the ambient Euclidean space that leave the polytope invariant setwise.

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The k-faces of a polytope form a lattice under inclusion. The elements of Π_0 are called vertices and the elements of Π_1 are called edges. Sometimes it is convenient to invent a single element of Π_{-1}, which we identify with the empty set. It is considered to have dimension -1.

Theorem (Borel–Tits; Satake; Casselman)

Suppose that the Coxeter group (W, S) acts by reflections as above, and that the subset $I(\mathbf{v}) = \{s \in S : s(\mathbf{v}) \neq \mathbf{v}\}$ consists of a single vertex. Then the W-orbits of k-dimensional faces of $\Pi(\mathbf{v})$ are in bijection with the connected k-subsets of S containing $I(\mathbf{v})$.

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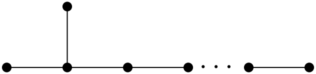
The theorem is true more generally; the hypothesis that $I(\mathbf{v})$ be a singleton is equivalent to the automorphism group being transitive on edges.
More is true: if in addition the Coxeter graph is a straight line and $I(\mathbf{v})$ is an endpoint, then the resulting polytope is regular. All regular polytopes can be constructed in this way.

Action on the half cube

Let us now return to the example of the Coxeter group $W = W(D_n)$ acting by reflections on $\mathbb{R}^n$, and define $\mathbf{v} = (2, 2, \ldots, 2)$.

D_n



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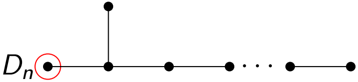

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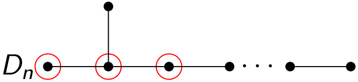

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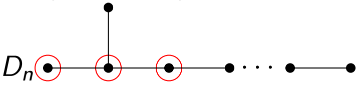

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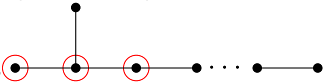

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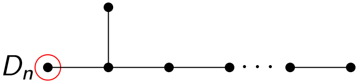
The orbit of $\mathbf{v}$ under the corresponding parabolic subgroup with $k = 3$ generators consists of $\mathbf{v}$ itself, together with the k points that differ from $\mathbf{v}$ only in coordinate positions 1 and $k + 1$, where they have an entry of -2. The convex hull of these points is a k-dimensional simplex.

The other type of face corresponds to a connected subgraph including both $s_{1'}$ and s_2.



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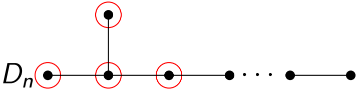

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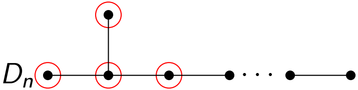

The orbit of $\mathbf{v}$ under the corresponding parabolic subgroup with $k = 4$ generators consists of $\mathbf{v}$ itself, together with the $2^{k-1} - 1$ points that differ from $\mathbf{v}$ in an even number of places in some or all of coordinate positions $1, \ldots, k$, where they have an entry of -2.

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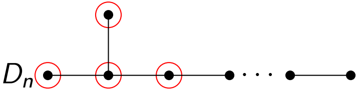

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In the half cube case, the chain groups C_k support natural linear actions of the group $W = W(D_n)$. The above classification of the faces of the half cube gives rise to an explicit description of the Coxeter group action on these chain groups, as follows.

Recall that the canonical W-orbit representatives of the faces of the half cube are in bijection with the connected subsets I of the Coxeter graph that contain the vertex $1'$.

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Translating this result into topology, we find that the action of W on C_k splits into submodules $C_{k,F}$, one for each type of face (i.e., half cube shaped or simplex shaped) corresponding to the orbit representative F.

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The character theory of Coxeter groups of classical type (A, B, D) is extremely well understood. It follows that, over $\mathbb{C}$, one can explicitly describe the characters of W acting on the chain groups C_k.

Homology representations

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Everything is homotopic to a wedge of spheres, probably of the same dimension.

Roughly speaking, this means that the space in question can be continuously morphed into a collection of spheres of the same dimension that are disjoint except that they have one point common to all of them.

The so-called Hawaiian earring shape shown below is reminiscent of a wedge of 1-dimensional spheres.

(Photo stolen without permission from Matt Macauley's Facebook post of April 1.)

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Theorem (G)

The subcomplex $C_{n,k}$ is homotopic to a wedge of spheres, all of dimension $k-1$. We define the number $b_{n,k}$ to be the number of spheres in the wedge.

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It follows from this result that the homology of $C_{n,k}$ is concentrated in degrees 0 and $k-1$, and the homology in degree 0 is 1-dimensional. The proof of the theorem gives a basis for the homology if $k > 3$.

The group $W = W(D_n)$ acts on $C_{n,k}$, because the removed faces constitute a union of W-orbits. Because W acts by continuous transformations, there is an induced action on the $(k-1)$-st homology of $C_{n,k}$.

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After tensoring over $\mathbb{C}$, this representation can be explicitly described, thanks to the Hopf trace formula. Basically, this result says that the alternating sum of the characters on the chain groups (which we know thanks to earlier results) is equal to the alternating sum of the characters on the homology representations (which can then be computed because only one of them is nontrivial).

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The combinatorial properties of $C_{n,k}$ include some interesting features of the Betti numbers $b_{n,k}$ (Sloane's sequence A119258).

The combinatorial properties of $C_{n,k}$ include some interesting features of the Betti numbers $b_{n,k}$ (Sloane's sequence A119258). They can be defined recursively by the conditions

$$b_{n,0} = b_{n,n} = 1$$

and, for $0 < k < n$,

$$b_{n,k} = 2b_{n-1,k} + b_{n-1,k-1}.$$

This information can be displayed using a Pascal-type triangle, as follows.

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$$
\begin{array}{ccccccccccccc}
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 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 1 & & & & \\
 & & & 1 & & 5 & & 7 & & 1 & & & \\
 & & 1 & & 7 & & 17 & & 15 & & 1 & & \\
 & 1 & & 9 & & 31 & & 49 & & 31 & & 1 & \\
1 & & 11 & & 49 & & 111 & & 129 & & 63 & & 1
\end{array}
$$

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The numbers $b_{n,k}$ show up in the following contexts:

(i) in the problem of finding, given n real numbers, a lower bound for the complexity of determining whether some k of them are equal;

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(iii) as the ranks of A-groups appearing in combinatorial homotopy theory;

(iv) as the number of nodes used by the Kronrod–Patterson–Smolyak cubature formula in numerical analysis; and

(v) (when $k = 3$) in engineering, as the number of three-dimensional block structures associated to n joint systems in the construction of stable underground structures.

Although some of the phenomena above are mysterious, the connection with the real hyperplane arrangement is more than numerology: the homology modules (at least over $\mathbb{C}$) are isomorphic as representations for the symmetric group, not just as vector spaces.

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Together with my student Jacob Harper, we are also investigating analogous properties of other highly symmetric nonregular polytopes, such as the hypersimplex. (The latter corresponds to a non-extremal vertex of the Coxeter graph of type A_n.)

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