The enumeration of fully commutative affine permutations

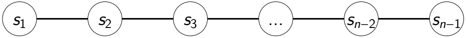
Christopher R. H. Hanusa
Queens College, CUNY

Joint work with Brant C. Jones, James Madison University

The affine permutations

(Finite) n-Permutations

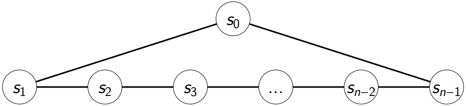
S_n has generators $\{s_1, \ldots, s_{n-1}\}$ and braid relations



Write elements in 1-line notation as a permutation of $\{1, 2, \ldots, n\}$.
Generators transpose a pair of entries: $s_i : (i) \leftrightarrow (i+1)$.
Example. $s_1 s_3 \in S_4$ is 2 1 4 3

Affine n-Permutations

$\widetilde{S_n}$ has generators $\{s_0, s_1, \ldots, s_{n-1}\}$ and braid relations



The affine permutations

Affine n-Permutations

Write elements in 1-line notation, as a permutation of $\mathbb{Z}$.

Generators transpose **infinitely many** pairs of entries:

s_i : **(i)** $\leftrightarrow$ **(i+1)** $\ldots (n+i) \leftrightarrow (n+i+1) \ldots (-n+i) \leftrightarrow (-n+i+1) \ldots$

In $\widetilde{S}_4$,	$\cdots w(-4)$	$w(-3)$	$w(-2)$	$w(-1)$	$w(0)$	$w(1)$	$w(2)$	$w(3)$	$w(4)$	$w(5)$	$w(6)$	$w(7)$	$w(8)$	$w(9)\cdots$
id	... -4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9 ...
s_1	... -4	-2	-3	-1	0	2	1	3	4	6	5	7	8	10 ...
s_0	... -3	-4	-2	-1	1	0	2	3	5	4	6	7	9	8 ...
$s_1 s_0$	... -2	-4	-3	-1	2	0	1	3	6	4	5	7	10	8 ...

★ Translational symmetry: $\widetilde{w}(i+n) = \widetilde{w}(i) + n$.

Therefore, $\widetilde{w}$ is defined by the window $[\widetilde{w}(1), \widetilde{w}(2), \ldots, \widetilde{w}(n)]$.

Example. In $\widetilde{S}_4$, $s_1 s_0 = [0, 1, 3, 6]$

Fully commutative elements

Definition. An element in a Coxeter group is **fully commutative** if it has only one reduced expression (up to commutation relations).

NO BRAIDS ALLOWED!

Example. In S_4, $s_1 s_2 s_3 s_1$ is **not fully commutative** because

$$s_1 s_2 s_3 s_1 \overset{\text{OK}}{=} s_1 s_2 s_1 s_3 \overset{\text{BAD}}{=} s_2 s_1 s_2 s_3$$

Question: **How many** fully commutative elements are there in S_n?

Answer: Catalan many! (Billey, Jockusch, Stanley, 1993; Knuth, 1973)

S_1: **1.** id

S_2: **2.** id, s_1

S_3: **5.** id, s_1, s_2, $s_1 s_2$, $s_2 s_1$

S_4: **14.** id, s_1, s_2, s_3, $s_1 s_2$, $s_2 s_1$, $s_2 s_3$, $s_3 s_2$, $s_1 s_3$, $s_1 s_2 s_3$, $s_1 s_3 s_2$, $s_2 s_1 s_3$, $s_3 s_2 s_1$, $s_2 s_1 s_3 s_2$

Enumerating fully commutative elements

Question: **How many** fully commutative elements are there in $\widetilde{S_n}$?

Answer: Infinitely many! (Even in $\widetilde{S}_3$.)

$$\text{id}, s_1, s_1 s_2, s_1 s_2 s_0, s_1 s_2 s_0 s_1, s_1 s_2 s_0 s_1 s_2, \ldots$$

Multiplying the generators cyclically does not introduce braids.

This is not the right question.

Enumerating fully commutative elements

Question: **How many** fully commutative elements are there in $\widetilde{S}_n$, with Coxeter length ℓ?

$$\text{In } \widetilde{S}_3: \quad \text{id}, \; \begin{matrix} s_0 \\ s_1 \\ s_2 \end{matrix}, \; \begin{matrix} s_0 s_1 & s_0 s_2 \\ s_1 s_0 & s_1 s_2 \\ s_2 s_0 & s_2 s_1 \end{matrix}, \; \begin{matrix} s_0 s_1 s_2 & s_0 s_2 s_1 \\ s_1 s_0 s_2 & s_1 s_2 s_0 \\ s_2 s_0 s_1 & s_2 s_1 s_0 \end{matrix}, \ldots$$

Question: Determine the coefficient of q^ℓ in the generating function

$$f_n(q) = \sum_{\widetilde{w} \in \widetilde{S}_n^{FC}} q^{\ell(w)}.$$

$$f_3(q) = 1q^0 + 3q^1 + 6q^2 + 6q^3 + \ldots$$

Answer: Consult your friendly computer algebra program.

DdddaaaaAAAAaaaaTTaaaaAA

Brant calls up and says: "Hey Chris, look at this data!"

$$f_3(q) = 1 + 3q + \mathbf{6}q^2 + \mathbf{6}q^3 + \mathbf{6}q^4 + \mathbf{6}q^5 + \cdots$$
$$f_4(q) = 1 + 4q + 10q^2 + \mathbf{16}q^3 + \mathbf{18}q^4 + \mathbf{16}q^5 + \mathbf{18}q^6 + \cdots$$
$$f_5(q) = 1 + 5q + 15q^2 + 30q^3 + 45q^4 + \mathbf{50}q^5 + \mathbf{50}q^6 + \mathbf{50}q^7 + \mathbf{50}q^8 + \cdots$$
$$f_6(q) = 1 + 6q + 21q^2 + 50q^3 + 90q^4 + 126q^5 + 146q^6 + \mathbf{150}q^7 + \mathbf{156}q^8 + \mathbf{152}q^9 + \mathbf{156}q^{10} + \mathbf{150}q^{11} + \mathbf{158}q^{12} + \mathbf{150}q^{13} + \mathbf{156}q^{14} + \mathbf{152}q^{15} + \mathbf{156}q^{16} + \mathbf{150}q^{17} + \mathbf{158}q^{18} + \cdots$$
$$f_7(q) = 1 + 7q + 28q^2 + 77q^3 + 161q^4 + 266q^5 + 364q^6 + 427q^7 + 462q^8 + 483q^9 + \mathbf{490}q^{10} + \mathbf{490}q^{11} + \mathbf{490}q^{12} + \mathbf{490}q^{13} + \cdots$$

Notice:

- The coefficients eventually repeat.

Goals: ★ Find a formula for the generating function $f_n(q)$.
★ Understand this periodicity.

Pattern Avoidance Characterization

Key idea: (**Green**, 2002)

$\widetilde{w}$ is fully commutative $\iff$ $\widetilde{w}$ is 321-avoiding.

Example. $[-4, -1, 1, 14]$ is **NOT** fully commutative because:

	$\cdots w(-4)$	$w(-3)$	$w(-2)$	$w(-1)$	$w(0)$	$w(1)$	$w(2)$	$w(3)$	$w(4)$	$w(5)$	$w(6)$	$w(7)$	$w(8)$	$w(9)\cdots$
$\widetilde{w}$	$\ldots$ 6	-8	-5	-3	**10**	-4	-1	**1**	14	**0**	3	5	18	4 $\ldots$

Game plan

Goal: Enumerate 321-avoiding affine permutations $\widetilde{w}$.

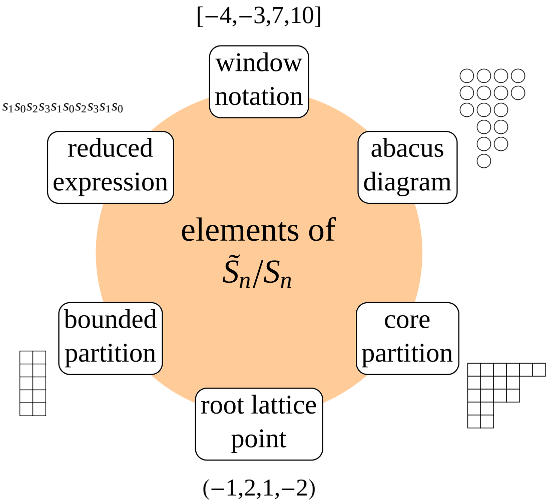
- Write $\widetilde{w} = w^0 w$, where $w^0 \in \widetilde{S_n}/S_n$ and $w \in S_n$.
  - w^0 **determines the entries;** w **determines their order.**

Example. For $\widetilde{w} = [-11, 20, -3, 4, 11, 0] \in \widetilde{S}_6$,

$$w^0 = [-11, -3, 0, 4, 11, 20] \text{ and } w = [1, 3, 6, 4, 5, 2].$$

- Determine which w^0 are 321-avoiding.
- Determine the finite w such that $w^0 w$ is still 321-avoiding

Combinatorial interpretations of $\widetilde{S}_n/S_n$



Combinatorial interpretations of $\widetilde{S}_n/S_n$

(James and Kerber, 1981)

Given $w^0 = [w_1, \ldots, w_n] \in \widetilde{S}_n/S_n$, we can interpret w^0 as:

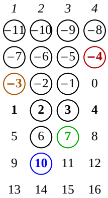
Abacus diagram

Place integers in n *runners*.
Circled: *beads*. Empty: *gaps*

Bijection: Given w^0, create an abacus where each runner has a lowest bead at w_i

Example:

$[-4, -3, 7, 10]$

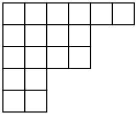


Core partition

An *n-core* is an integer partition with no n-ribbons.

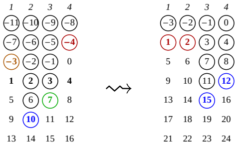
Bijection: Read the boundary steps from the abacus:

Bead = vertical; Gap = horiz.



Normalized abacus and 321-avoiding criterion for $\widetilde{S_n}/S_n$

We use a *normalized* abacus diagram; shifts all beads so that the first gap is in position $n+1$; this map is invertible.



Theorem. (H–J '09) Given a normalized abacus for $w^0 \in \widetilde{S_n}/S_n$, where the last bead occurs in position i,

w^0 is fully commutative $\iff$ lowest beads in runners only occur in $\{1, \ldots, n\} \cup \{i-n+1, \ldots, i\}$

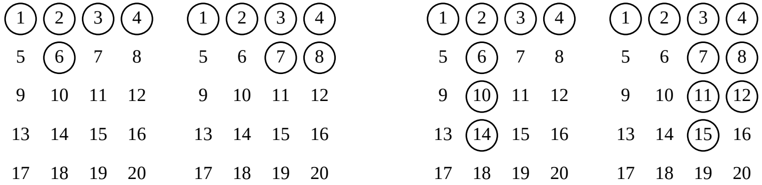
Idea: Lowest beads in runners $\leftrightarrow$ entries in base window.

w(-n+1)	w(-n+2)	...	w(-1)	w(0)	w(1)	w(2)	...	w(n-1)	w(n)	w(n+1)	w(n+2)	...	w(2n-1)	w(2n)
lo	lo	...	hi	hi	lo	lo	...	hi	hi	lo	lo	...	hi	hi
lo	lo	med	hi	**hi**	lo	lo	**med**	hi	hi	**lo**	lo	med	hi	hi

Long versus short elements

Partition $\widetilde{S_n}$ into long and short elements:

Short elements	**Long elements**
Lowest bead in position $i \leq 2n$	Lowest bead in position $i > 2n$
Finitely many	Come in infinite families
Hard to count	**Easy to count**
	Explain the periodicity



Enumerating long elements

For long elements $\widetilde{w} \in \widetilde{S_n}$, the base window for w^0 is $[a, a, \dots, a, b, b, \dots, b]$ where $1 \leq a \leq n$, and $n+2 \leq b$.

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16
17	18	19	20

Question: Which permutations $w \in S_n$ can be multiplied into a w^0?

- We can not invert any pairs of a's, nor any pairs of b's. (Would create a 321-pattern with an adjacent window)
- Only possible to *intersperse* the a's and the b's.

How many ways to intersperse (k) a's and $(n-k)$ b's? $\binom{n}{k}$

BUT: We must also keep track of the *length* of these permutations. This is counted by the q-binomial coefficient: $\begin{bmatrix} n \\ k \end{bmatrix}_q$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q)_n}{(q)_k (q)_{n-k}}, \text{ where } q_n = (1-q)(1-q^2)\cdots(1-q^n)$$

Enumerating long elements

After we:

- Enumerate by length all possible w^0 with (k) a's and $(n-k)$ b's.
- Combine the Coxeter lengths by $\ell(\widetilde{w}) = \ell(w^0) + \ell(w)$.

Then we get:

Theorem. (H–J '09) For a fixed $n \geq 0$, the generating function by length for *long* fully commutative elements $\widetilde{w} \in \widetilde{S}_n^{FC}$ is

$$\sum q^{\ell(\widetilde{w})} = \frac{q^n}{1-q^n} \sum_{k=1}^{n-1} \begin{bmatrix} n \\ k \end{bmatrix}_q^2 .$$

Periodicity of fully commutative elements in $\widetilde{S}_n$

Corollary. (H–J '09) The coefficients of $f_n(q)$ are eventually periodic with period dividing n.

When n is prime, the period is 1: $a_i = \frac{1}{n}\left(\binom{2n}{n} - 2\right)$.

Proof. For i sufficiently large, all elements of length i are long. Our generating function is simply some polynomial over $(1 - q^n)$:

$$\frac{q^n}{1-q^n}\sum_{k=1}^{n-1}\begin{bmatrix} n \\ k \end{bmatrix}_q^2 = \frac{P(q)}{1-q^n} = P(q)(1 + q^n + q^{2n} + \cdots)$$

When n is prime, an extra factor of $(1 + q + \cdots + q^{n-1})$ cancels;

$$\frac{1}{1-q}\left[\frac{q^n}{1+q+\cdots+q^{n-1}}\sum_{k=1}^{n-1}\begin{bmatrix} n \\ k \end{bmatrix}_q^2\right]$$

As suggested by a referee, we know that $a_i = P(1) = \frac{1}{n}\sum_{k=1}^{n-1}\binom{n}{k}^2$.

Short elements are hard

For short elements $\widetilde{w} \in \widetilde{S_n}$, the base window for w^0 is $[a, \ldots, a, b, \ldots, b, c, \ldots, c]$, and there is more interaction:

1	2	3	4
5	6	7	8
9	10	11	12

No a can invert with an a or b. No c can invert with a b or c.

- Count $\widetilde{w}$ where some a intertwines with some c.
- Count $\widetilde{w}$ w/o intertwining and 0 descents in the b's.
- Count $\widetilde{w}$ w/o intertwining and 1 descent in the b's.
  - Not so hard to determine the acceptable finite permutations w.
  - Such as $\sum_{M\geq 0} x^{L+M+R} \sum_{\mu=1}^{M-1} \left(\begin{bmatrix} M \\ \mu \end{bmatrix}_q - 1 \right) \begin{bmatrix} L+\mu \\ \mu \end{bmatrix}_q \begin{bmatrix} R+M-\mu \\ M-\mu \end{bmatrix}_q$
- Count $\widetilde{w}$ w/o intertwining and 2 descents in the b's.
- Count $\widetilde{w}$ which are finite permutations. (Barcucci et al.)
  - Solve functional recurrences (Bousquet-Mélou)
  - Such as $D(x, q, z, s) =$ $N(x, q, z, s) + \frac{xqs}{1-qs}\big(D(x, q, z, 1) - D(x, q, z, qs)\big) + xsD(x, q, z, s)$

Future Work

- Extend to $\widetilde{B_n}$, $\widetilde{C_n}$, and $\widetilde{D_n}$
  - Develop combinatorial interpretations (Wait 10 minutes...)
  - 321-avoiding characterization?
- Heap interpretation of fully commutative elements
  - Can use Viennot's heaps of pieces theory
  - Better bound on periodicity
- More combinatorial interpretations for $\widetilde{W}/W$
  - What do you know?

Thank you!

Slides available: `people.qc.cuny.edu/chanusa` > `Talks`

Christopher R. H. Hanusa and Brant C. Jones.
The enumeration of fully commutative affine permutations
European Journal of Combinatorics. Vol 31, 1342–1359. (2010)

Christopher R. H. Hanusa and Brant C. Jones.
Abacus models for parabolic quotients of affine Coxeter groups

Anders Björner and Francesco Brenti.
Combinatorics of Coxeter Groups, Springer, 2005.