Homology representations arising from a hypersimplex

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What is a Hypersimplex?

Definition

A **polytope** is the convex hull of a finite set of points in $\mathbb{R}^n$. Let $J(n,k)$ be the polytope equal to the convex hull of

$$\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x_j \in \{0,1\} \text{ and } \sum x_j = k\}.$$

For any $n, k \in \mathbb{Z}$ such that $1 \leq k \leq n-1$ and $n \geq 2$, this polytope is called a **hypersimplex**.

Relationship to Coxeter Groups

- Let the symmetric group S_n act on $\mathbb{R}^n$ by permuting the coordinates.
- Then $J(n, k)$ is equal to the convex hull of the group orbit $S_n \cdot v_0$ where $v_0 = (1_1, \ldots, 1_k, 0_{k+1}, \ldots, 0_n) \in \mathbb{R}^n$.

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The Convex Hulls of Coxeter Group Orbits

Theorem (Casselman, 1997)

Let $\delta = \{\alpha \in \Delta \mid (\alpha, v) \neq 0\}$ for some $v \in V$ and let $W_\kappa = \langle s \in S \mid \alpha_s \in \kappa\rangle$ for some $\kappa \subset \Delta$. Suppose that $v \in V$ lies in the fundamental domain $D = \{\lambda \in V \mid (\lambda, \alpha) \geq 0$ for all $\alpha \in \Delta\}$, then the map taking κ to the convex hull F_κ of $W_\kappa \cdot v$ is a bijection between the δ-connected subsets of Δ and the set of W-representatives of the faces of the convex hull of $W \cdot v$.

Application of Casselman's Theorem

- The vertices, or 0-faces, of $J(n,k)$ are given by

$$V = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x_j \in \{0,1\} \text{ and } \sum x_j = k\}.$$

- For each set $\{j, j+1, \ldots, j+i\}$ such that $1 \leq j \leq k \leq j+i \leq n$, there is an orbit of i-faces with representative given by the convex hull of

$$\{(1_1, \ldots, 1_{j-1}, x_j, \ldots, x_{j+i}, 0_{j+i+1}, \ldots, 0_n)\} \subset V.$$

- Every face appears in exactly one of these orbits and each orbit fixes a unique number of 0's and 1's.

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Diagram for the Faces of J(18,8)

	9,0	8,0	7,0	6,0	5,0	4,0	3,0	2,0	1,0	0,0
	9,1	8,1	7,1	6,1	5,1	4,1	3,1	2,1	1,1	0,1
	9,2	8,2	7,2	6,2	5,2	4,2	3,2	2,2	1,2	0,2
	9,3	8,3	7,3	6,3	5,3	4,3	3,3	2,3	1,3	0,3
	9,4	8,4	7,4	6,4	5,4	4,4	3,4	2,4	1,4	0,4
	9,5	8,5	7,5	6,5	5,5	4,5	3,5	2,5	1,5	0,5
	9,6	8,6	7,6	6,6	5,6	4,6	3,6	2,6	1,6	0,6
10,8	9,7	8,7	7,7	6,7	5,7	4,7	3,7	2,7	1,7	0,7

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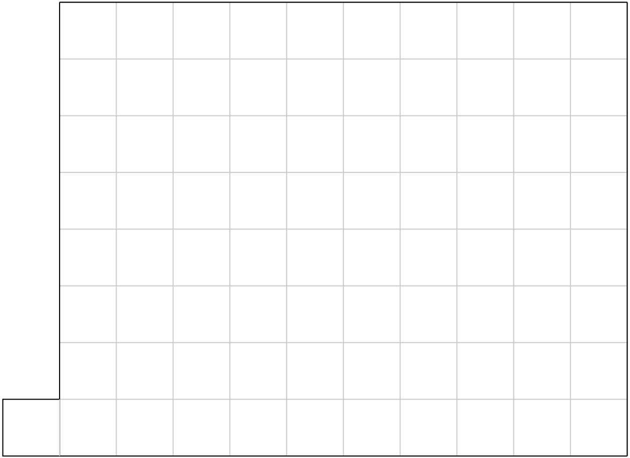

Homology Groups of J(n,k)

- It is a standard result that the faces of a polytope form a regular CW complex.
- Given the CW complex K associated with $J(n,k)$, there exists a corresponding cellular chain complex.

$$\cdots \longrightarrow C_{i+1} \xrightarrow{\partial_{i+1}} C_i \xrightarrow{\partial_i} C_{i-1} \longrightarrow \cdots$$

The C_i are free abelian groups with a basis in one-to-one correspondence with the i-faces of $J(n,k)$ and the maps ∂_i are defined by letting

$$\partial_i(e_\tau) = \sum_{\sigma \in X^{(i-1)}} [\tau : \sigma] e_\sigma .$$

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Homology Groups of J(n,k)

- The i^{th} homology group of K is then

$$H_i(K) = \ker(\partial_i)/\mathrm{Im}(\partial_{i+1}).$$

- It should be noted that we can modify K by considering $\emptyset$ as a unique cell of dimension -1. The resulting cellular chain complex then leads to the reduced homology groups.

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Goal

- There is an action of S_n on the faces of $J(n,k)$.
  - This induces an action on the chain groups C_i.
  - This further induces an action on the homology groups.
- The goal is to identify the representations that arise when we instead act on subcomplexes of K whose reduced homology groups are concentrated in a single degree.

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Definition

Let K be a finite regular CW complex. A **discrete vector field** on K is a collection of pairs of cells (K_1, K_2) such that

(i) K_1 is a face of K_2 of codimension 1 and

(ii) every cell of K lies in at most one such pair.

Definition

If V is a discrete vector field on a regular CW complex K, a **V-path** is a sequence of cells

$$a_0, b_0, a_1, b_1, a_2, \dots, b_r, a_{r+1}$$

such that for each $i = 0, \dots, r$, a_i and a_{i+1} are each a codimension 1 face of b_i, each of the pairs (a_i, b_i) belongs to V (hence a_i is matched with b_i), and $a_i \neq a_{i+1}$ for all $0 \leq i \leq r$. If $r \geq 0$, we call the V-path **nontrivial** and if $a_0 = a_{r+1}$, we call the V-path **closed**.

Hasse Diagram of K

- Let V be a discrete vector field on a regular CW complex K.
- Consider the set of cells of K together with the empty cell $\emptyset$, which we consider as a cell of dimension -1.
- This gives us a partially ordered set ordered under inclusion.
- We can create a directed Hasse diagram, $H(V)$, from this by pointing all of the edges towards the larger cell and then reversing the direction of any edge in which the smaller cell is matched with the larger cell.

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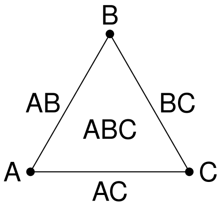
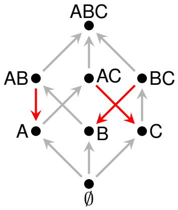
Example of H(V)

Example

Consider the hypersimplex $J(3,1)$. Label the vertices by A, B, and C, the edges by AB, AC, and AB, and the unique 2-face by ABC. Let $V = \{(A, AB), (B, BC), \text{ and } (C, AC)\}$ be a discrete vector field.



$J(3,1)$



$H(V)$

- If $H(V)$ has no directed cycles then we say that V is an **acyclic (partial) matching** of the Hasse diagram of K.

Theorem (Forman, 2002)

Let V be a discrete vector field on a regular CW complex K.

(i) *There are no nontrivial closed V-paths if and only if V is an acyclic matching of the Hasse diagram of K.*

(ii) *Suppose that V is an acyclic partial matching of the Hasse diagram of K in which the empty set is unpaired. Let u_p denote the number of unpaired p-cells. Then K is homotopic to a CW complex with exactly u_p cells of dimension p for each $p \geq 0$.*

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Notation for Faces of J(n,k)

- Let S be a sequence made up of 0's, 1's, and *'s of length n. Define $F(S)$ to be the face of $J(n,k)$ equal to the convex hull of the set of vertices of this polytope whose coordinates written out in a sequence differ from S only where S has a *. Let $S(0)$ and $S(1)$ denote the number of 0's and 1's in the sequence S respectively.

Example

If $S = 10 * * * 010$ with $n = 8$ and $k = 3$, then $F(S)$ is the face equal to the convex hull of the following points:

- (1,0,1,0,0,0,1,0)
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Construction 1

Let $F(S)$ be a face of $J(n,k)$. Match $F(S)$ with $F(S')$ where S' is obtained from S by doing one of the following replacements to S:

1. If $S(1) \leq k-1$ and there is a 1 to the right of the rightmost * in S, then replace the rightmost 1 with a *.
2. If $S(1) \leq k-2$ and there is no 1 right of the rightmost * in S, then replace the rightmost * with 1. (inverse of 1)
3. If $S(1) = k-1$, there is no 1 to the right of rightmost *, and there is a 0 to the left of the leftmost * in S, then replace the leftmost 0 with *.
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Example of Matching

Example

Consider $J(n, k)$ and so $n = 8$ and $k = 3$, then:

- F(0100*0*1) and F(0100*0**) are matched by (1) and (2)
- F(1000*1*0) and F(1*00*1*0) are matched by (3) and (4)

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Diagram of J(18,8) with Matching 1

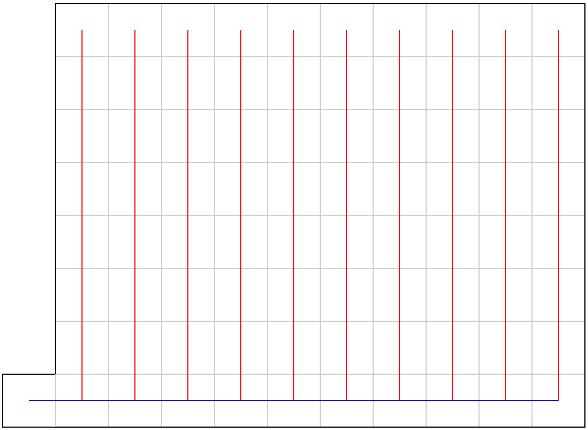
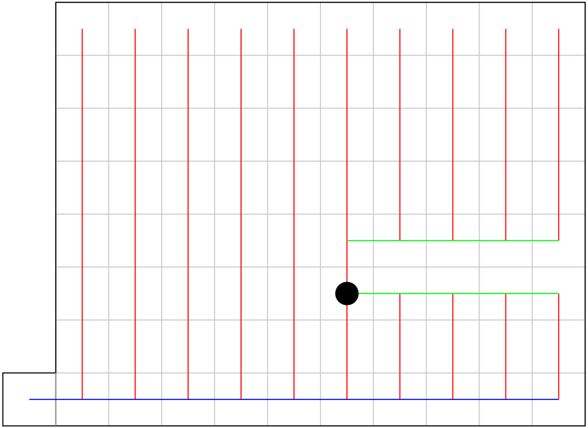


Diagram of J(18,8) with Matching 2



Theorem (Harper)

The two matchings described are both acyclic.

Lemma

If a subcomplex K' of K has unmatched faces in a single dimension (in any matching) then K' can be described completely by one or more of the following properties:

(i) *K' contains every face $F(S)$ of dimension at most d, $0 \leq d \leq n-2$.*

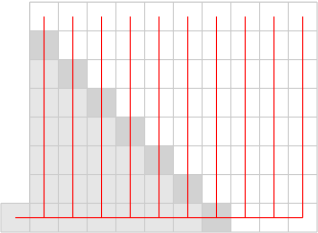
(ii) *K' contains every face $F(S)$ such that S has at least j 1's, $1 \leq j \leq k-1$.*

(iii) *K' contains every face $F(S)$ such that S has at least i 0's, $1 \leq i \leq n-k-1$.*

(i)

(ii)

(i) & (ii)



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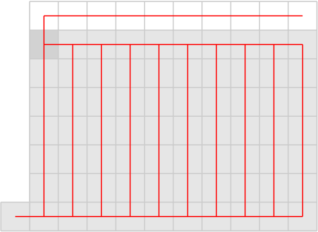
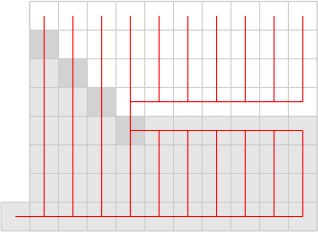


(ii)



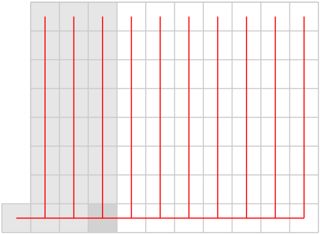
(i) & (ii)

(iii)

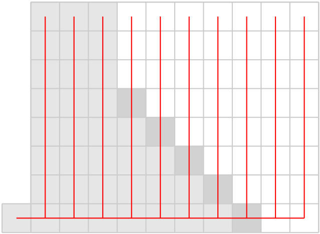
(i) & (iii)

(ii) & (iii)

(i), (ii), & (iii)



(iii)



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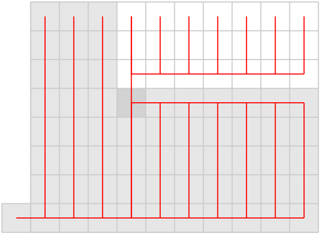
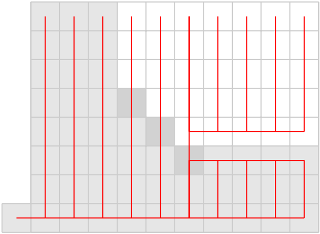


(ii) & (iii)



(i), (ii), & (iii)

Theorem (Harper)

Let K be the CW complex obtained from $J(n,k)$. The subcomplexes K' of K described in the previous lemma are the only ones whose reduced homology groups are concentrated in a single degree. Furthermore, the reduced homology groups of K' are free abelian of rank equal to the number of unmatched faces in that dimension.

Theorem (Hopf trace formula)

Let K be a finite complex with (integral) chain groups $C_p(K)$ and homology groups $H_p(K)$. Let $T_p(K)$ be the torsion subgroup of $H_p(K)$. Let $\phi : C_p(K) \longrightarrow C_p(K)$ be a chain map, and let ϕ_* be the induced map on homology. Then we have that

$$\sum_p (-1)^p \operatorname{tr}(\phi, C_p(K)) = \sum_p (-1)^p \operatorname{tr}(\phi_*, H_p(K)/T_p(K)).$$

Definition

Let $a, b \in \mathbb{Z}$ such that $a \geq b$. Then we define f_1, f_2, f_3, and f_4 in the following way:

(i) $$f_1(a,b) = \sum_{c=0}^{b-1} \chi^{[a+c,\ b-c,\ 1^{n-a-b}]}$$

(ii) $$f_2(a,b) = \sum_{c=0}^{b} \chi^{[a+c,\ b+1-c,\ 1^{n-a-b-1}]}$$

(iii) $$f_3(a,b) = \sum_{c=0}^{b-1} \chi^{[a+1+c,\ b-c,\ 1^{n-a-b-1}]}$$

(iv) $$f_4(a,b) = \sum_{c=0}^{b} \chi^{[a+1+c,\ b+1-c,\ 1^{n-a-b-2}]}$$

where $\chi^{[a,\ b+1,\ 1^{n-a-b-1}]} = 0$ if $a = b$ and $f_1(a,b) = f_3(a,b) = 0$ if $b = 0$. Also, if $a < b$ then define $f_i(a,b) = f_i(b,a)$ for $1 \leq i \leq 4$.

Theorem (Harper)

Let K' be a subcomplex of K such that the reduced homology groups of K' are concentrated in a single degree d. Let

$$U = \{\big(S(0), S(1)\big) \mid F(S) \notin K',\ \text{but } F(S) \text{ is matched to a face in } K'\}.$$

Let (a_0, b_0) be the element of U such that $a_0 < a$ for all other $(a, b) \in U$. Then the character of the representation of S_n on the d^{th} homology of the complex K', denoted by $\chi(H_d(K'))$, is the following:

$$f_4(a_0, b_0) + \sum_{\substack{(a,b)\in U\setminus\{(a_0,b_0)\} \\ a\geq b}} f_2(a, b) + f_4(a, b) + \sum_{\substack{(a,b)\in U\setminus\{(a_0,b_0)\} \\ a<b}} f_3(a, b) + f_4(a, b).$$