Towards a Cyclic Version of Matsumoto's Theorem

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Coxeter groups

Definition

A *Coxeter system* consists of a group W (called a Coxeter group) generated by a set S of involutions with presentation

$$W = \langle S \mid \; s^2 = 1, \quad (st)^{m(s,t)} = 1 \rangle$$

where $m(s,t) \geq 2$ for $s \neq t$.

Since s and t are involutions, the relation $(st)^{m(s,t)} = 1$ can be rewritten as

$$\begin{array}{lcll}
m(s,t) = 2 & \Longrightarrow & st = ts & \left.\vphantom{x}\right\} \quad \textbf{short braid relations} \\
m(s,t) = 3 & \Longrightarrow & sts = tst & \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} \quad \textbf{long braid relations} \\
m(s,t) = 4 & \Longrightarrow & stst = tsts & \\
 & \vdots & &
\end{array}$$

A Coxeter element $w \in W$ is the product of the generators in some order.

Coxeter graphs

Definition

A Coxeter system can be encoded by a unique Coxeter graph Γ with

- vertex set S
- edges $\{s, t\}$ for each $m(s, t) \geq 3$, labeled with $m(s, t)$.

- Edges correspond to non-commuting pairs of generators.
- W is irreducible if Γ is connected.

Example ($W = \widetilde{A}_5$).

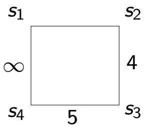


Figure: Coxeter graph for $W = \langle s_1, s_2, s_3, s_4 \mid s_1^2, s_2^2, s_3^2, s_4^2, (s_1 s_3)^2, (s_2 s_4)^2, (s_1 s_2)^3, (s_2 s_3)^4, (s_3 s_4)^5 \rangle$.

Reduced expressions

A word $s_{x_1} s_{x_2} \cdots s_{x_m} \in S^*$ is an expression for w if it is equal to w when considered as an element of W.

If m is minimal, it is a reduced expression for w, and $m := \ell(w)$ is its length.

Example

Consider the expression $s_1 s_3 s_2 s_1 s_2$ for an element $w \in W(A_3)$. Note that

$$s_1 s_3 s_2 s_1 s_2 = s_1 s_3 s_1 s_2 s_1 = s_3 s_1 s_1 s_2 s_1 = s_3 s_2 s_1 .$$

Therefore, $s_1 s_3 s_2 s_1 s_2$ is not reduced, and $\ell(w) = 3$.

A word is cyclically reduced if every cyclic shift of it is a reduced expression.

The short braid relations (e.g., $st = ts$) generate an equivalence relation on S^*. Two expressions for the same group element that differ *only* by short braid relations are in the same commutation class.

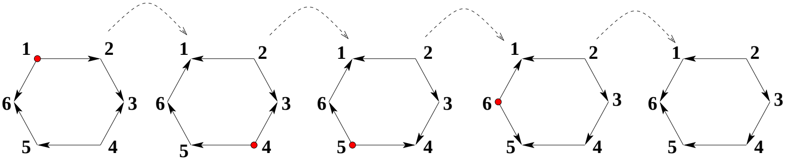
An element $w \in W$ is fully commutative (FC) if *all* of its reduced expressions lie in the same commutation class.

Source-to-sink operations

There is a canonical bijection between the set $C(W)$ of Coxeter elements and the set $\mathrm{Acyc}(\Gamma)$ of acyclic orientations of Γ.

- A cyclic shift (conjugation) corresponds to a source-to-sink operation (or "click").

Example:



$s_1 s_2 s_4 s_5 s_3 s_6$ | $s_4 s_2 s_5 s_3 s_6 s_1$ | $s_2 s_5 s_3 s_6 s_1 s_4 = s_5 s_2 s_3 s_6 s_1 s_4$ | $s_2 s_3 s_6 s_1 s_4 s_5 = s_6 s_2 s_3 s_1 s_4 s_5$ | $s_2 s_3 s_1 s_4 s_5 s_6$

Figure: Conjugating a Coxeter element

This generates an *equivalence relation* $\sim_\kappa$ on $\mathrm{Acyc}(\Gamma)$ (and on $C(W)$).

Clearly, if $c \sim_\kappa c'$ for two Coxeter elements, then c and c' are conjugate. But does the converse hold?

Conjugacy of Coxeter elements

Definition

An element $w \in W$ is logarithmic if $\ell(w^k) = k \cdot \ell(w)$ for all $k \in \mathbb{N}$.

Theorem (Speyer, 2007)

If W is an infinite irreducible Coxeter group, Coxeter elements are logarithmic.

Speyer's proof strongly relies on the *source-to-sink* equivalence relation.

Soon after, H. Eriksson and K. Eriksson proved the following.

Theorem (Eriksson, 2008)

Two Coxeter elements $c, c' \in \mathrm{C}(W)$ are conjugate if and only if $c \sim_\kappa c'$.

However, many more elements than just the Coxeter elements are logarithmic. And the "source-to-sink" property holds for a more general class as well.

Beyond Coxeter elements

What is the proper generalization of the Erikssons' result, beyond just Coxeter elements?

▶ If an element is fully commutative (FC), then we can associate it with a canonical acyclic directed graph (more formally, a heap).

The source-to-sink property holds for Coxeter elements because:

- Coxeter elements are fully commutative;
- Every cyclic shift of a Coxeter element is fully commutative.

This motivates us to define the following:

Definition

An element $w \in W$ is cyclically fully commutative (CFC) if every cyclic shift of every reduced expression for w is FC.

▶ In summary, the FC elements are those whose reduced expressions avoid long braid relations. The CFC elements are those whose reduced expressions, when "written as a circular word," still avoid long braid relations.

The word and conjugacy problems

A classic result by Matsumoto and/or Tits elegantly solves the word problem for Coxeter groups.

Theorem (Matsumoto / Tits)

Any two reduced expressions for the same group element differ by braid relations.

Motivated by the Erikssons' theorem, we ask the following:

Do two cyclically reduced expressions of conjugate elements differ by a sequence of braid relations and cyclic shifts?

This is a cyclic version of Matsumoto's theorem. It would be to the conjugacy problem what Matsumoto's theorem is to the word problem.

The Erikssons' theorem can be rephrased to say that this "Cyclic Version of Matsumoto's Theorem" holds for Coxeter elements.

Cyclic version of Matsumoto's theorem. . . FAIL.

"Unfortunately," the cyclic version of Matsumoto's theorem (CVMT) fails in general.

There is a really easy counterexample. In $W(A_2)$, s_1 and s_2 do not differ by cyclic shifts, but

$$s_1 s_2 (s_1) s_2 s_1 = s_1 s_2 s_1 s_2 s_1 = s_1 s_1 s_2 s_1 s_1 = s_2 .$$

Despite this, it seems to "usually be true." So, when does the CVMT hold? It seems like there ought to be a simple elegant answer. Understanding which elements are logarithmic would be a good start.

Definition

An element $w \in W$ is torsion-free if the minimal parabolic subgroup containing it has no finite factors.

Equivalently, every connected component of the induced Coxeter graph $\Gamma[\mathrm{supp}(w)]$ must induce an infinite Coxeter group.

Clearly, only torsion-free elements stand a chance of being logarithmic.

The logarithmic property

Question

Which elements are logarithmic?

- Being cyclically reduced, and torsion-free are two easy necessary conditions.

For Coxeter elements, they are sufficient as well. (Though this sounds easy, the proof of it took up an entire *Proc. AMS* paper; Speyer 2009.)

Theorem (Ernst, Green, M–)

Let W be a Coxeter group with no odd bond strengths greater than 3. Then a CFC element $w \in W$ is logarithmic iff it is cyclically reduced and torsion-free.

We are working on dropping the restriction on the bond strenghts. This is harder than it looks! Richard Green calls it the "Rabbit Hole of Death."

The conjugacy problem

In trying to prove that CFC elements are logarithmic, we have the combinatorial tools that come along with the source-to-sink equivalence.

Furthermore, we expect that we can use these tools to prove the cyclic version of Matsumoto's theorem for these elements.

Conjecture

Let $w \in \mathrm{CFC}(W)$ be torsion-free and cyclically reduced. Then

(a) *w is logarithmic;*

(b) *the cyclic version of Matsumoto's theorem holds for $\mathrm{cl}(w)$.*

Also, this surely holds for more than just the CFC elements. But how do we handle the presense of braid relations?

The root poset

Let $\Phi = \Phi^+ \sqcup \Phi^-$ be the set of all roots of a Coxeter group W.

We can represent roots as vectors in $\mathbb{R}^n$ and partially order them by $\leq$ componentwise, to get the root poset.

Each generator $s_i \in S$ acts on Φ by reflection:

$$\vec{z} \xmapsto{s_i} \vec{z} + \sum_{j=1}^{n} 2\cos\left(\frac{\pi}{m_{ij}}\right) z_j e_i \ .$$

▶ In summary, multiplication by s_i flips the sign of the i^{th} entry and adds each neighboring state z_j weighted by $2\cos(\pi/m_{ij}) \geq 1$.

We can represent this as a linear map $F_i : \mathbb{R}^n \longrightarrow \mathbb{R}^n$:

$$F_i : (z_1, \ldots z_{i-1}, z_i, z_{i+1}, z_n) \longmapsto \left(z_1, \ldots, z_{i-1}, z_i + \sum_{j=1}^{n} 2\cos\left(\frac{\pi}{m_{ij}}\right) z_j, z_{i+1}, \ldots, z_n\right) .$$

The root automaton

Consider a word $w = s_{x_1} s_{x_2} s_{x_3} \cdots s_{x_{m-1}} s_{x_m} \in S^*$.

Start at the vector $\vec{e}_{x_1} \in \Phi^+$ (a positive root), and follow the paths labeled $s_{x_2}, s_{x_3}, s_{x_4}, \ldots$.

If s_{x_k} is the first instance of crossing over to the negative roots Φ^-, then w is not a reduced expression, and moreover,

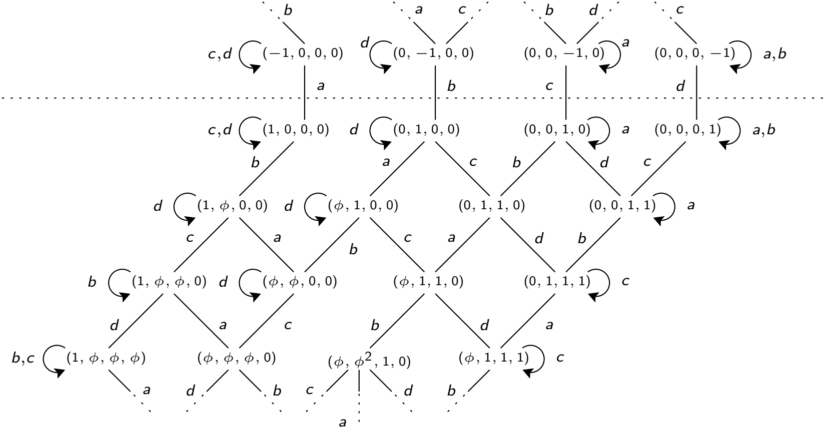
$$s_{x_1} s_{x_2} s_{x_3} \cdots s_{x_{k-1}} s_{x_k} = s_{x_2} s_{x_3} \cdots s_{x_{k-1}} .$$

- Thus, the root automaton detects reduced words.

Let $\vec{r}(w)$ denote the root

$$F_{x_m} \circ F_{x_{m-1}} \circ \cdots \circ F_{x_3} \circ F_{x_2} [\vec{e}_{x_1}] .$$

Example: The root automaton of type H_4.



A Seifert–van-Kampen approach to Coxeter groups

Let Z be a cut-set of Γ, separating $S_1, S_2 \subset S$ with $S = S_1 \cup S_2$ and $S_1 \cap S_2 = Z$. Let $G = W(S_1 \setminus Z)$ and $H = W(S_2 \setminus Z)$ be the corresponding parabolic subgroups.

Proposition

Suppose that $w \neq 1$ and $\ell(g) \geq \ell(h)$. Then

(i) $\vec{r}(ghw) = \vec{r}(gw)$

(ii) $\vec{r}(wgh) = \vec{r}(wg) + \vec{r}(wh) - \vec{r}(w)$.

We can build reduced words in W from reduced words in the standard parabolic subgroups $W(S_1)$ and $W(S_2)$.

Conjecture

Suppose that $g_1 z_1 g_2 z_2 \cdots g_{k-1} z_{k-1} g_k$ is reduced in $W(S_1)$ and $h_1 z_1 h_2 z_2 \cdots h_{k-1} z_{k-1} h_k$ is reduced in $W(S_2)$. For each $i = 1, \ldots, k$, let a_i be either g_i, h_i, or $g_i h_i$. Then the word

$$w = a_1 z_1 a_2 z \cdots a_{k-1} z_{k-1} a_k$$

is reduced in W.

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