Geometrical enumeration of certain factorisations of a Coxeter element in finite reflection groups

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LaCIM — UQÀM

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Invariant theory of W (geometry of the discriminant Δ_W) $\leftrightarrow$ Combinatorics of the noncrossing partition lattice of W (factorizations of a Coxeter element)

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N.B. : the structure doesn't depend on the choice of the Coxeter element (conjugacy) $\rightsquigarrow$ write $\mathrm{NC}(W)$.

Coxeter-Catalan combinatorics

Number of multichains in NC(W)

Suppose W irreducible of rank n, and let c be a Coxeter element. The number of multichains $w_1 \preccurlyeq_R w_2 \preccurlyeq_R \dots \preccurlyeq_R w_p \preccurlyeq_R c$ is the Fuß-Catalan number of type W

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$\rightsquigarrow$ how to understand this formula uniformly ?

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- *Bad news* : we obtain much more complicated formulas.
- *Good news* : we can interpret some of them (and even refine them) geometrically ; in particular for $p = n$ or $n - 1$.

Submaximal factorizations of a Coxeter element

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Theorem (R.)

Let Λ be a conjugacy class of elements of length 2 of $\text{NC}(W)$. Call submaximal factorizations of c of type Λ the block factorizations containing $n-2$ reflections and one element (of length 2) in the conjugacy class Λ. Then, their number is:

$$|\text{FACT}^{\Lambda}_{n-1}(c)| = \frac{(n-1)!\ h^{n-1}}{|W|} \deg D_{\Lambda} \ ,$$

where D_{Λ} is a homogeneous polynomial constructed from the geometry of the discriminant hypersurface of W.

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$$\begin{array}{ccccc} L_0 \in \mathcal{L} & \leftrightarrow & W_0 \in \mathrm{PSG}(W) & \leftarrow & c_0 \text{ parabolic Coxeter element} \\ \mathrm{codim}(L_0) & = & \mathrm{rk}(W_0) & = & \ell_R(c_0) \end{array}$$

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$$\rightsquigarrow \text{isomorphism:} \quad \begin{array}{ccc} V/W & \xrightarrow{\sim} & \mathbb{C}^n \\ \bar{v} & \mapsto & (f_1(v), \ldots, f_n(v)). \end{array}$$

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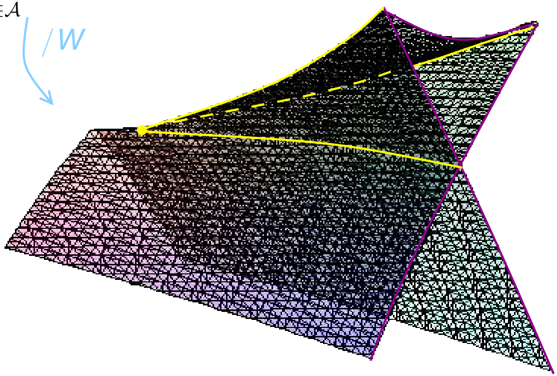

hypersurface $\mathcal{H}$ (discriminant) $\subseteq W\backslash V \simeq \mathbb{C}^3$

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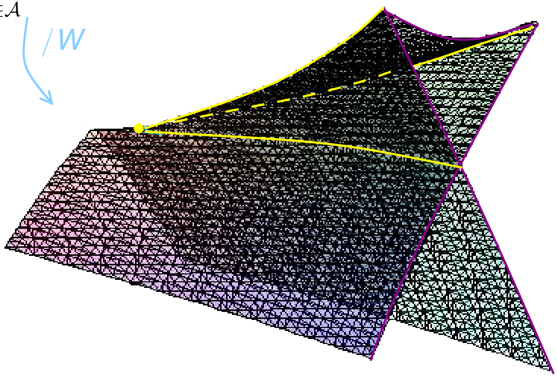
Example of $W = A_3$: stratification of the discriminant

$$\bigcup_{H \in \mathcal{A}} H \subseteq V$$

$/W$



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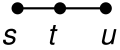

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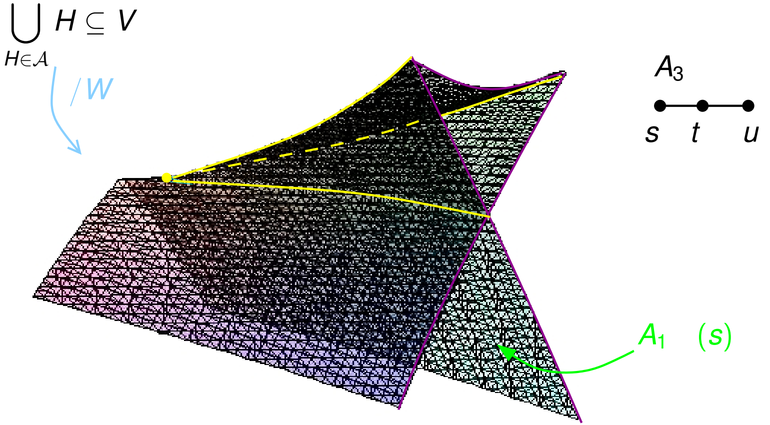

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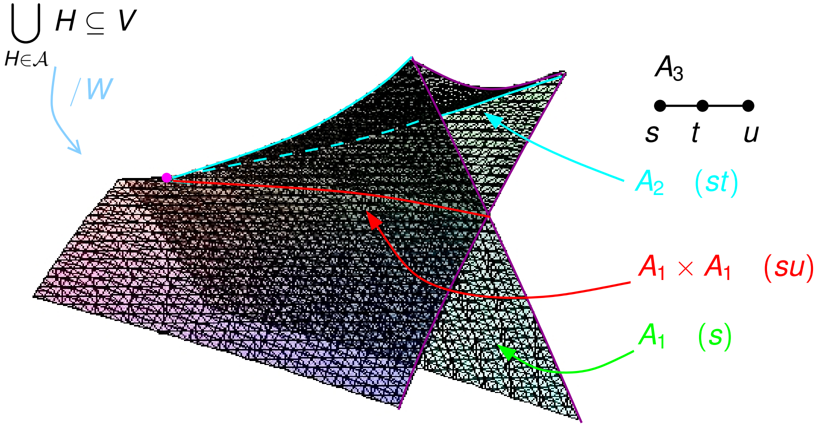

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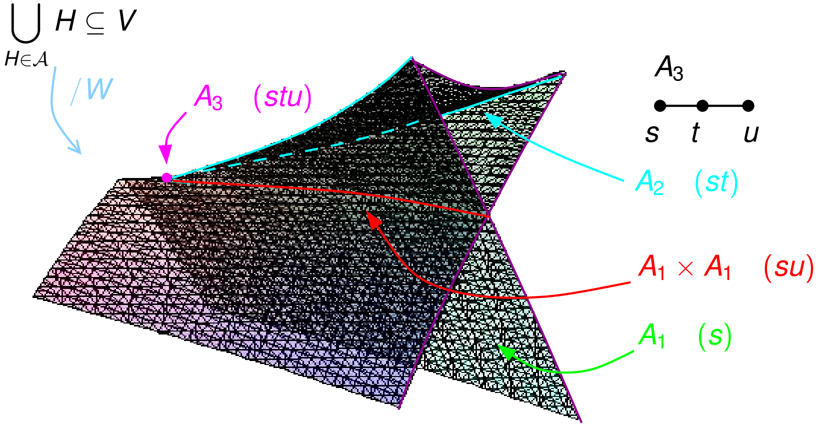

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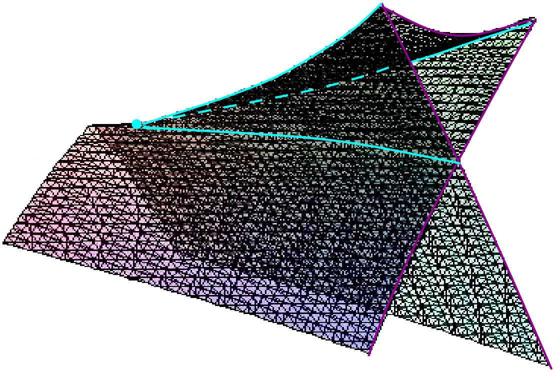
Definition

The bifurcation locus of Δ_W (w.r.t. f_n) is the hypersurface of $\mathbb{C}^{n-1}$:

$$\mathcal{K} := \{D_W = 0\}$$

Example of A_3: bifurcation locus $\mathcal{K}$



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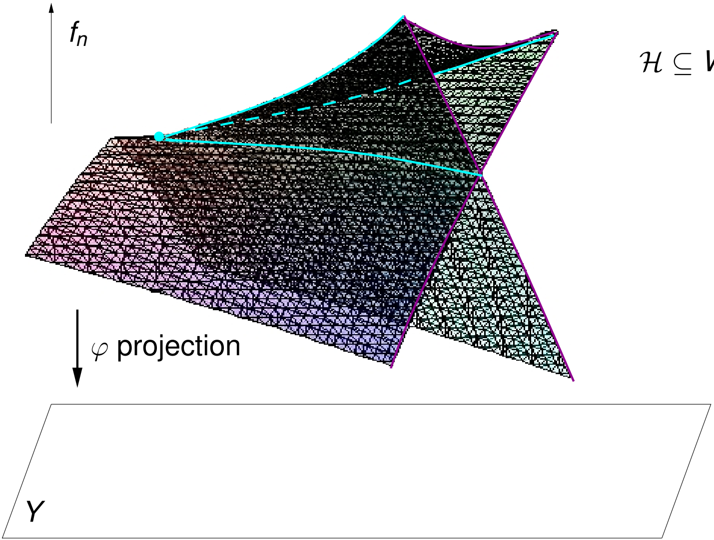

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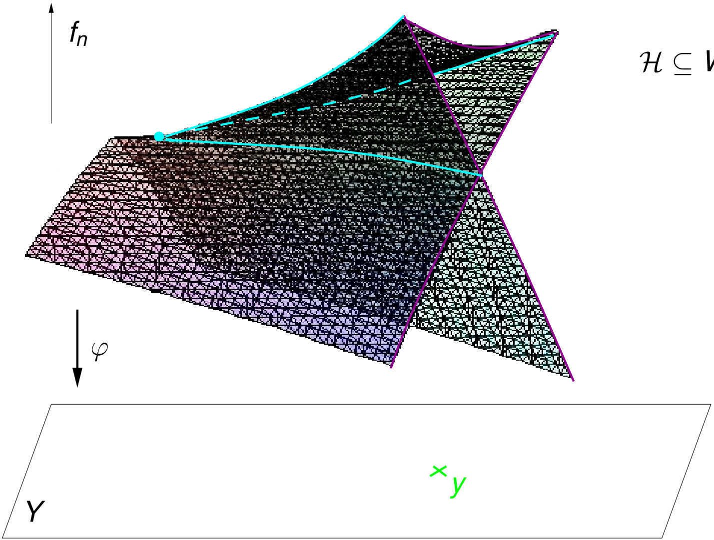

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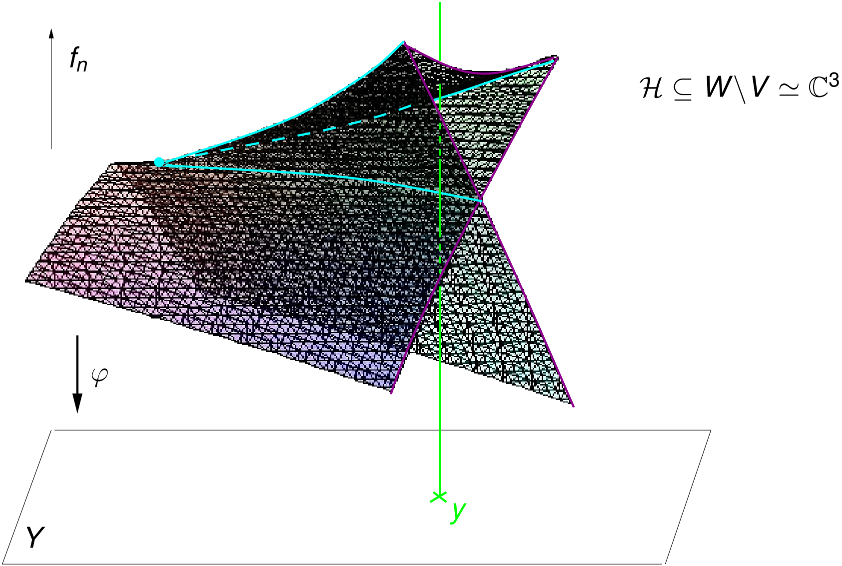

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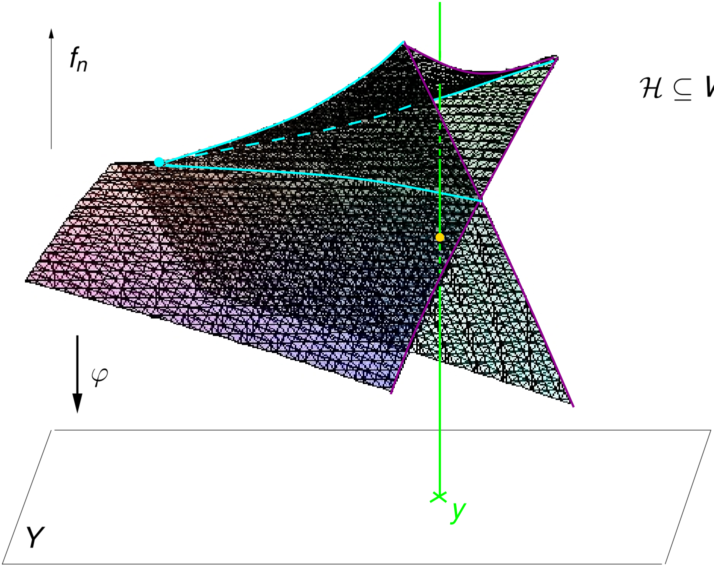

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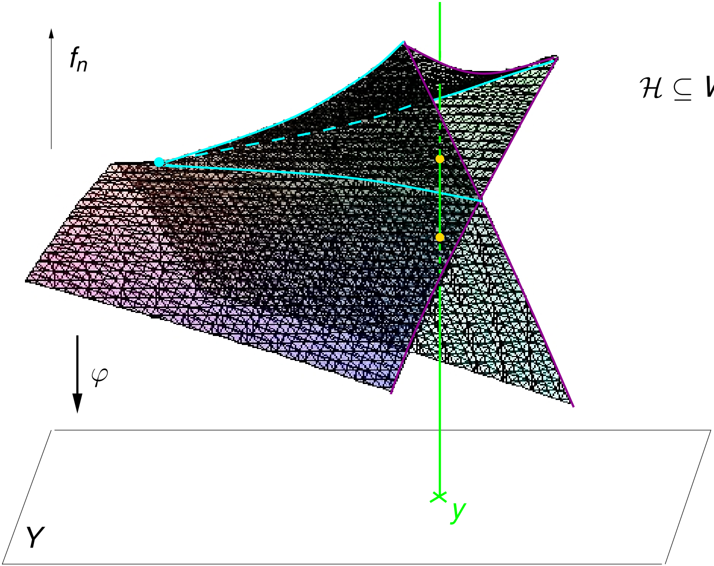

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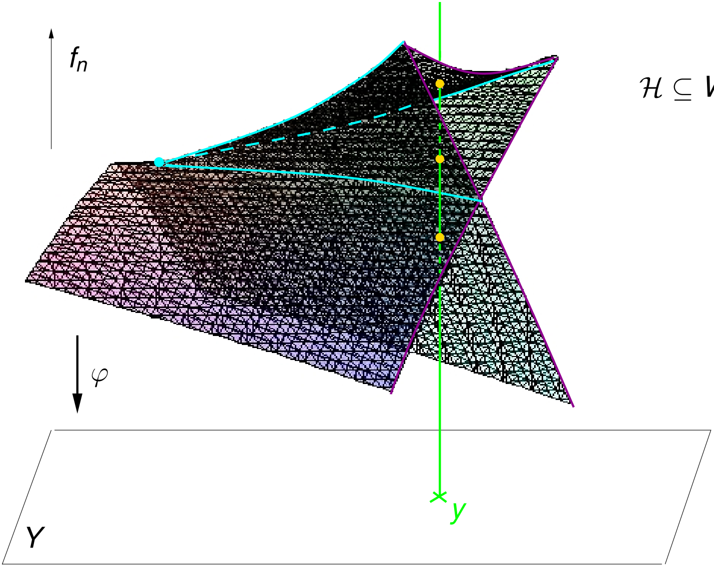

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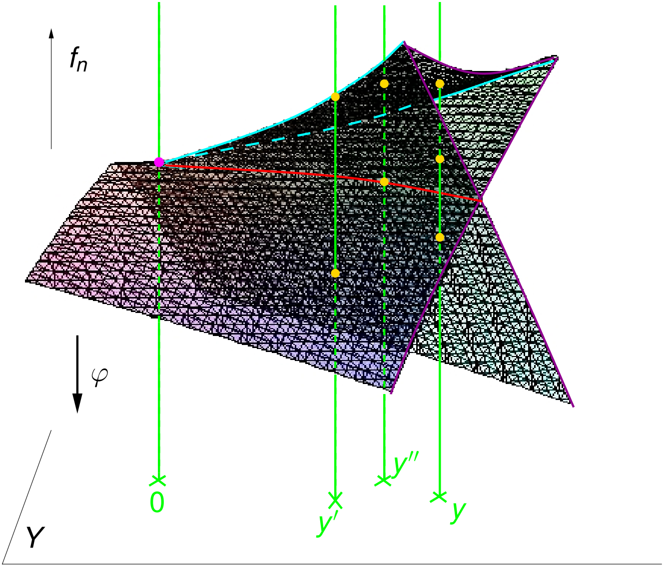

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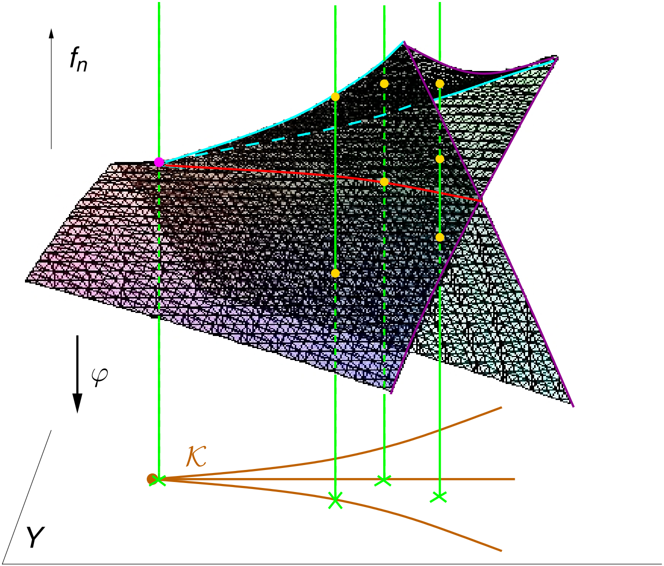

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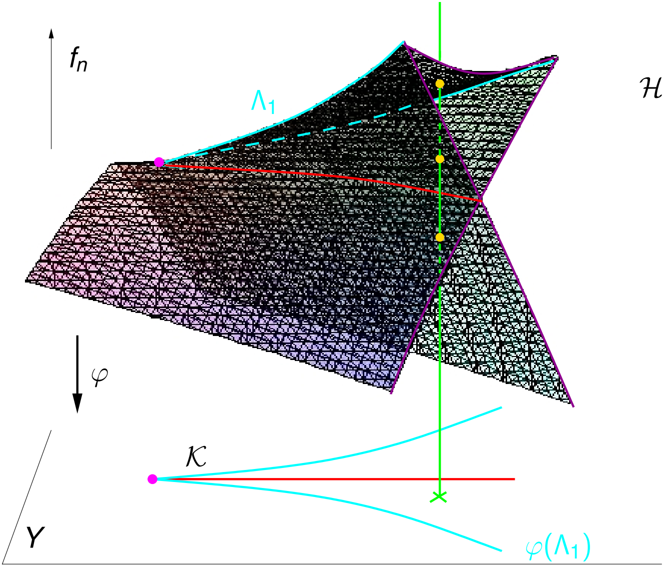

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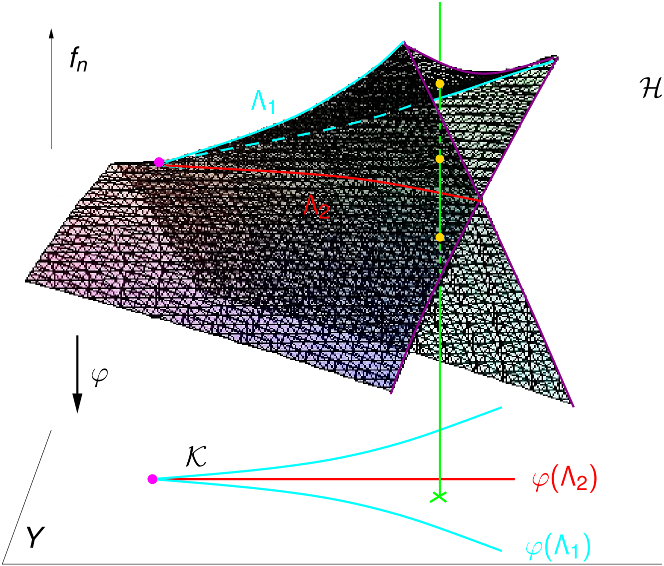

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Corollary Proof End

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▸ Return to thm

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Corollary

*The number of **block factorisations of a Coxeter element** c **in** $n-1$ **factors** is:*

$$|\operatorname{FACT}_{n-1}(c)| = \frac{(n-1)!\ h^{n-1}}{|W|}\left(\frac{(n-1)(n-2)}{2}h + \sum_{i=1}^{n-1} d_i\right),$$

where $d_1, \ldots, d_n = h$ are the invariant degrees of W.

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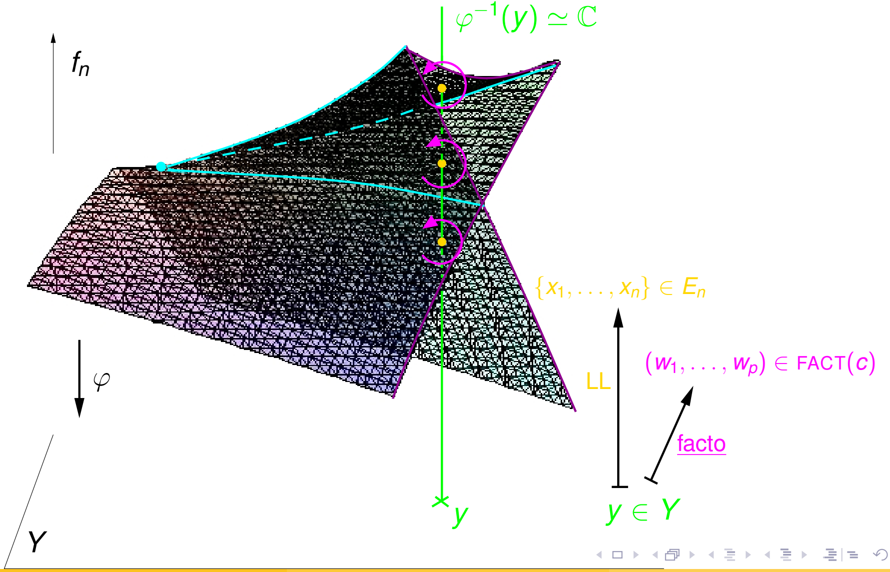
Lyashko-Looijenga morphism and topological factorisations

▸ More details ▸ Return to thm ▸ End



Some ingredients of the proof

See picture End

- Lyashko-Looijenga morphism LL:

$$y \in Y = \mathrm{Spec}\, \mathbb{C}[f_1, \dots, f_{n-1}] \mapsto \text{multiset of roots of } \Delta_W(y, f_n).$$

- Construction of topological factorisations: [Bessis, R.]

$$\text{facto} : Y \to \mathrm{FACT}(c)\ .$$

- Fundamental property that the product map:

$$Y \xrightarrow{\mathrm{LL} \times \text{facto}} E_n \times \mathrm{FACT}(c)$$

  is injective, and its image is the set of "compatible" pairs.
  In other words, the map facto induces a bijection between any fiber $\mathrm{LL}^{-1}(\omega)$ and the set of factorisations of same "composition" as ω.
- Consequently, we can use some algebraic properties of LL to obtain cardinalities of certain fibers, and deduce enumeration of certain factorisations.