A uniform bijection between nonnesting and noncrossing partitions

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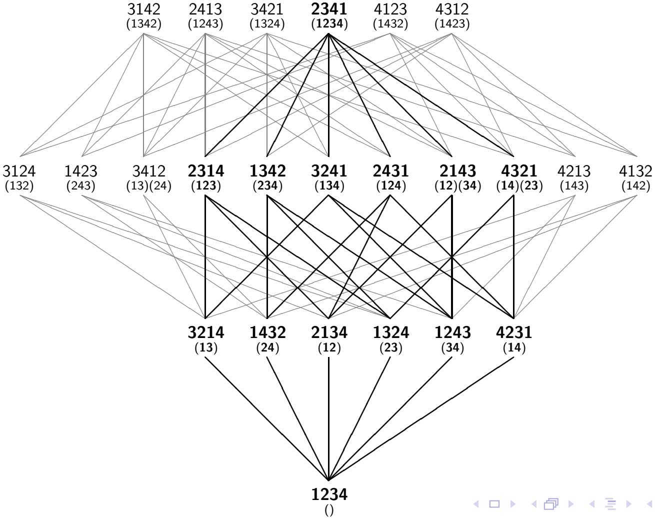
What am I going to talk about?

I will talk about

- **noncrossing partitions**, which are certain elements in a finite Coxeter group W
  - they generalize noncrossing set partitions
- **nonnesting partitions**, which are antichains in the root poset associated to a finite, crystallographic Coxeter group W
  - they generalize nonnesting set partitions
- **connections** between noncrossing and nonnesting partitions
- a **uniform bijection** between both and some of its properties
- the proof of its existence, which is type-dependent and which leads to **new combinatorics** in the classical types

Noncrossing partitions

Let W be a finite Coxeter group with reflections T and a Coxeter element c. The **noncrossing partition lattice** $NC(W,c)$ is the interval $[\mathbf{1}, c]$ in the **Cayley graph** of (W, T).

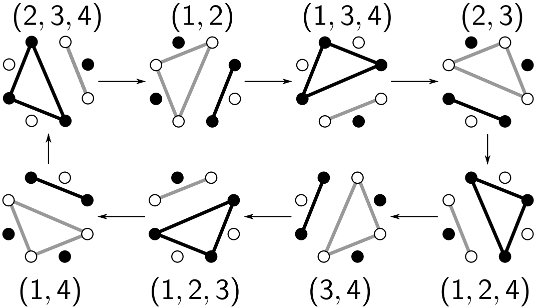


A cyclic action on noncrossing partitions

Kreweras defined a cyclic action on $NC(W, c)$ given by

$$\mathcal{K} : \sigma \mapsto c\sigma^{-1}.$$

Observe that K^2 is the conjugation with the Coxeter element c.



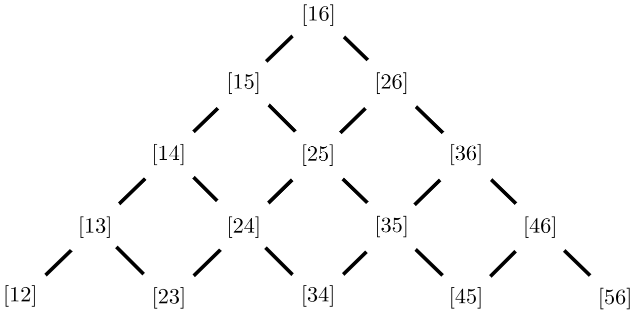
Nonnesting partitions

Let W be a finite crystallographic Coxeter group with root system Φ. Let $\Delta \subseteq \Phi^+ \subseteq \Phi$ be collections of simple and positive roots.

Definition

The **root poset structure** on Φ^+ is given by

$$\alpha \leq \beta \Leftrightarrow \beta - \alpha \in \mathbb{N}\Delta.$$



$$[ij] = e_i - e_j \in \mathbb{R}^n$$

A cyclic action on nonnesting partitions

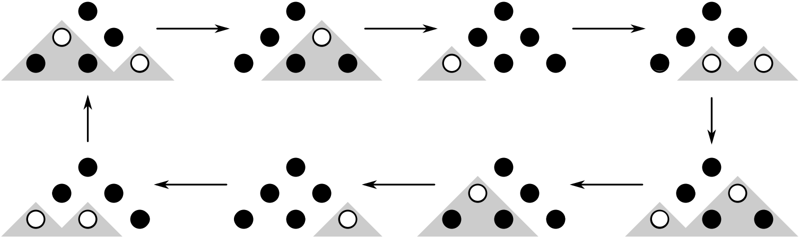
Definition

A **nonnesting partition** is an antichain in Φ^+. Let

$$NN(W) := \{A \subseteq \Phi^+ : A \text{ antichain}\}.$$

Panyushev defined a cyclic action on $NN(W)$ given by

$$\mathcal{P} : A \mapsto \min\{\beta \in \Phi^+ : \beta \notin \mathcal{I}(A)\}.$$



Connections between noncrossing and nonnesting partitions

- Noncrossing and nonnesting partitions are both counted by the Catalan numbers for finite Coxeter groups,

$$|NC(W,c)| = |NN(W)| = \mathrm{Cat}(W) := \prod_{i=1}^{\ell} \frac{d_i + h}{d_i},$$

  - for nonnesting partitions this formula is proved uniformly,
  - for noncrossing partitions this formula is proved case-by-case,
- several bijections between both are known in the classical types but no type-independent bijection was known,
- both where conjectured to exhibit a "cyclic sieving" with the same q-Catalan numbers for the Kreweras and the Panyushev map.

A uniform connection between $NN(W)$ and $NC(W, c)$

Let

- $S = L \sqcup R$ be a bipartition of the simple reflections such that the elements in L and in R pairwise commute,
- $\Delta = \Delta_L \sqcup \Delta_R$ be the bipartition of the simple roots,
- $c = c_L c_R$ be the bipartite Coxeter element.

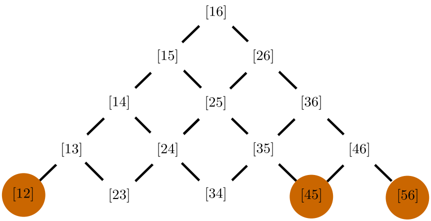
Theorem (AST 2011)

There exists a **unique bijection** ψ *from* $NN(W)$ *to* $NC(W, c)$ *satisfying the following three properties:*

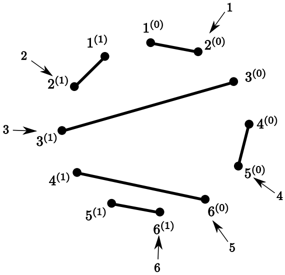
- $\psi(\Delta_L) = \mathbf{1}$, (initial condition)
- $\psi \circ \mathrm{Pan} = \mathrm{Krew} \circ \psi$, $(\mathrm{Pan} = \mathrm{Krew})$
- $\psi(I) = c_L\big|_{L \setminus \mathrm{supp}(I)} \quad \psi\big|_{\mathrm{supp}(I)}(I)$. (parabolic recursion)

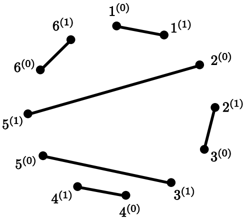
A case-by-case realization of the bijection

$$\psi_A : NN(\mathfrak{S}_n) \stackrel{\sim}{\longrightarrow} NC(\mathfrak{S}_n, c = (1, 2, \ldots, n))$$



$\mapsto$





$\mapsto (235) \in NC(\mathfrak{S}_6, c)$

Implications of the constructions

Theorem (AST 2011, conjectured by Panyushev)

The Panyushev map $\mathcal{P}$ has the following properties:

- $\mathcal{P}^{2h}$ *acts as the identity on* $NN(W)$,
- $\mathcal{P}^{h}$ *acts as* $-\omega_0$ *on* $NN(W)$,
- *the average number of positive roots in an antichain in a Panyushev orbit equals half the rank of the group,*

$$\frac{1}{|\mathcal{O}|}\sum_{A\in\mathcal{O}}|A| = \ell/2,$$

- $\mathcal{P}$ *exhibits a "cyclic sieving" on nonnesting partitions.*