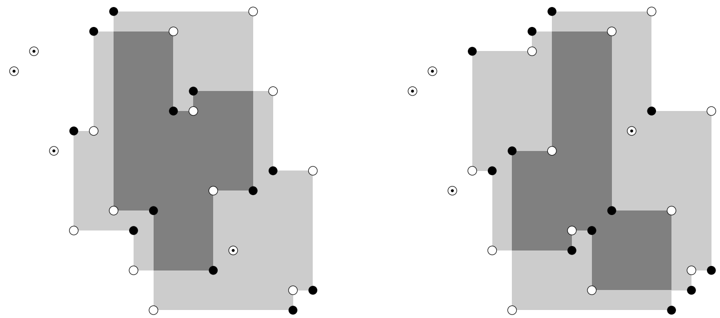
On the μ-coefficient of Kazhdan-Lusztig polynomials

Greg Warrington — University of Vermont



AMS Spring Eastern Sectional Meeting

College of the Holy Cross, Worcester, MA

Élie Cartan's 142nd Birthday

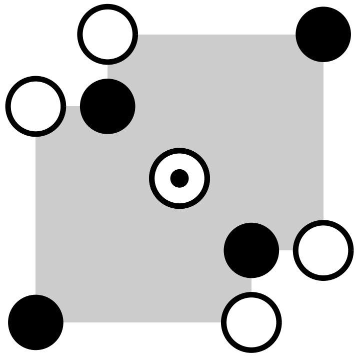
Kazhdan-Lusztig polynomials

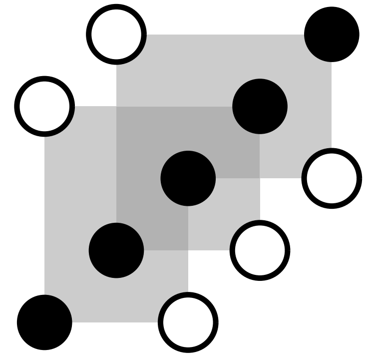
Define $P_{x,w}(q)$ for $x, w \in S_n$.

Facts

- $P_{x,x}(q) = 1$.
- $P_{x,w}(q) = 0$ when $x \not\leq w$.
- $\deg P_{x,w}(q) \leq \frac{\ell(w)-\ell(x)-1}{2}$ when $x \leq w$.

Kazhdan-Lusztig polynomials





$w = 34201, \ \ell(w) = 8$ $\qquad$ $v = 34012, \ \ell(v) = 6$

$x = 03214, \ \ell(x) = 3$ $\qquad$ $u = 01234, \ \ell(v) = 0$

$P_{x,w}(q) = 1 + q^2$ $\qquad$ $P_{u,v}(q) = 1 + 2q$

When is $P_{x,w}(q)$ of max possible degree?

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Define

$$\mu(x,w) = \left[q^{\frac{\ell(w)-\ell(x)-1}{2}} \right] P_{x,w}(q)$$

$$\mu[x,w] = \max\{\mu(x,w), \mu(w,x)\}.$$

The recursion

$$P_{x,w}(q) = q^{1-c} P_{xs,ws}(q) + q^{c} P_{x,ws}(q) - \sum_{\substack{z \leq ws \\ zs < z}} \mu(z, ws) q^{\frac{\ell(w)-\ell(z)}{2}} P_{x,z}(q)$$

$0-1$ conjecture

Is $\mu(x, w)$ always 0 or 1?

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Theorem [McLarnan-W '03]

No.

$0 - 1$ conjecture

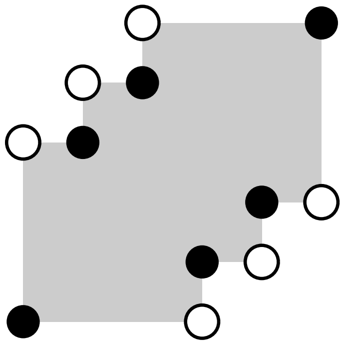
Is $\mu(x, w)$ always 0 or 1?

Theorem [McLarnan-W '03]

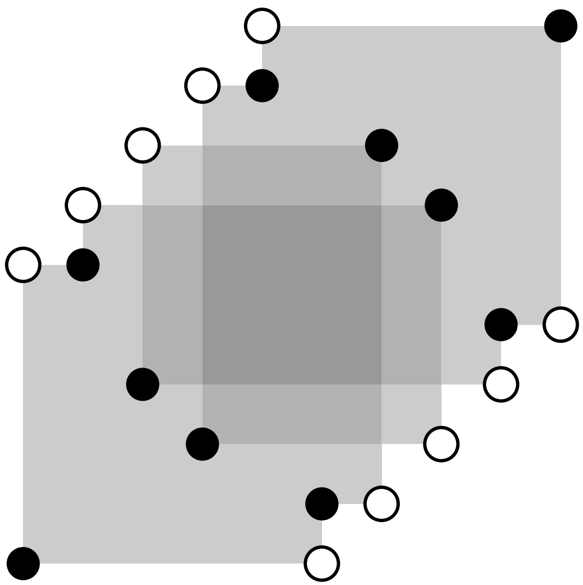
No.

Not even close.

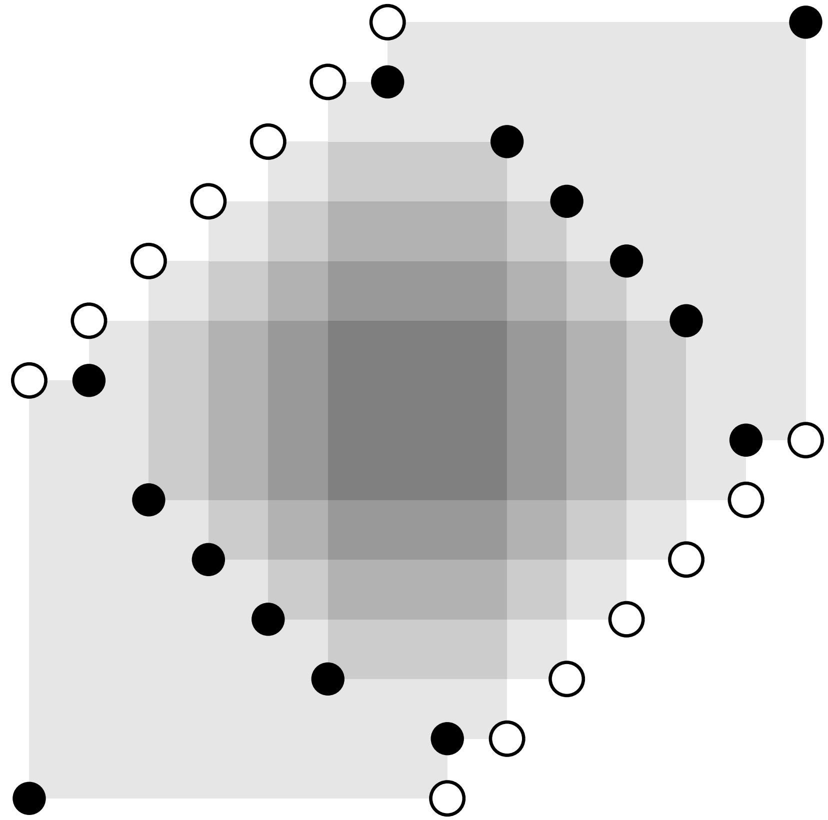
A family of counterexamples



$1 + 4q + q^2$



$1 + 16q + 75q^2 + 124q^3 + 58q^4 + 4q^5$



$\dots + 620q^7 + 19q^8$

A Sequence

$a_n = 4a_{n-1} + 3a_{n-2}$	0,1,4,19,88,409,1900, 8827,41008,190513,...
Trees of diameter 8	1,4,19,66,219,645, 1813,4802,12265,...
Powers of $\sqrt{19}$ rounded down	1,4,19,82,361,1573, 6859,29897,130321,...

Possible values

What values are attained by $\boldsymbol{\mu}(\boldsymbol{x}, \boldsymbol{w})$?

Current knowledge

Set $M(n) = \{\mu(x,w) : x, w \in S_n\} \setminus \{0\}$.

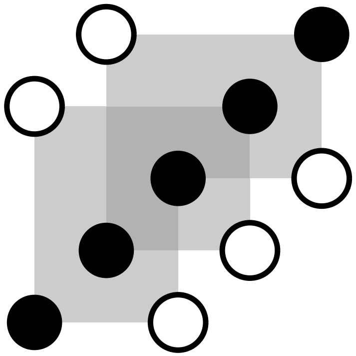
Theorem [W '10] We have

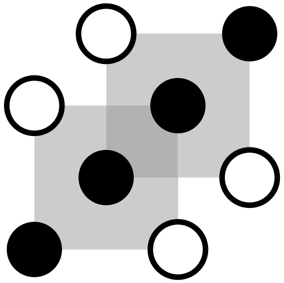
- $M(10) = \{1, 4, 5\}$
- $M(11) = \{1, 3, 4, 5, 18, 24, 28\}$
- $M(12) \supseteq M(11) \cup \{2, 6, 7, 8, 23, 25, 26, 27, 158, 163\}$
- $\max_{x,w \in S_{13}} \mu(x,w) \geq 796$

Main Problem

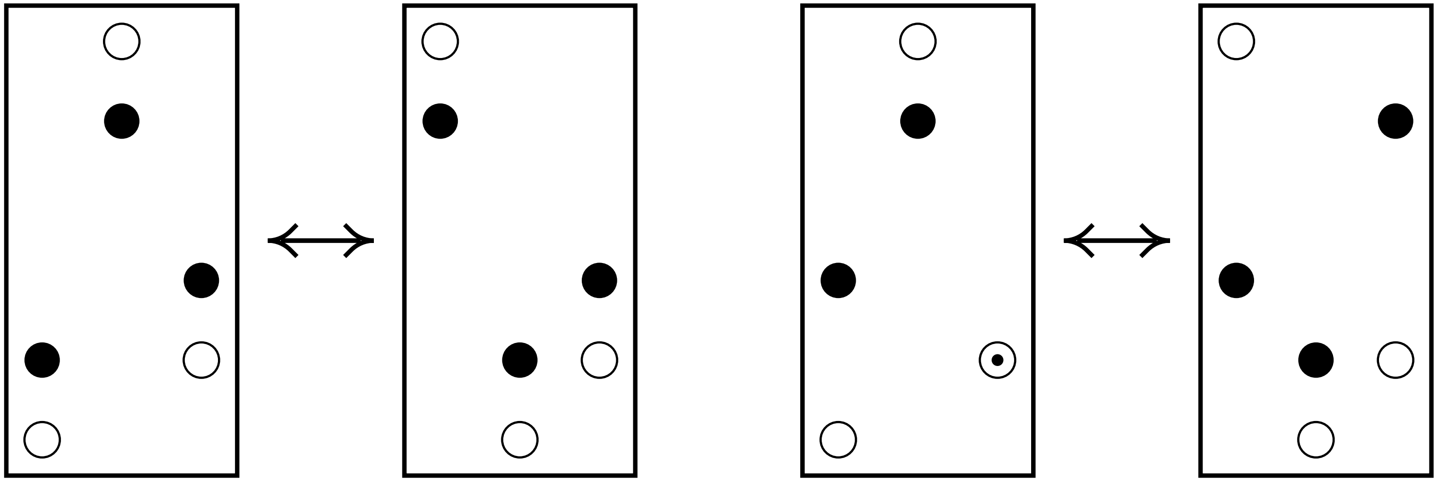
$\|\boldsymbol{S}_{10} \times \boldsymbol{S}_{10}\|$	13,168,189,440,000
$\#\{\boldsymbol{x} \leq \boldsymbol{w}\}$	$\sim$ 800,000,000,000

Many are zero



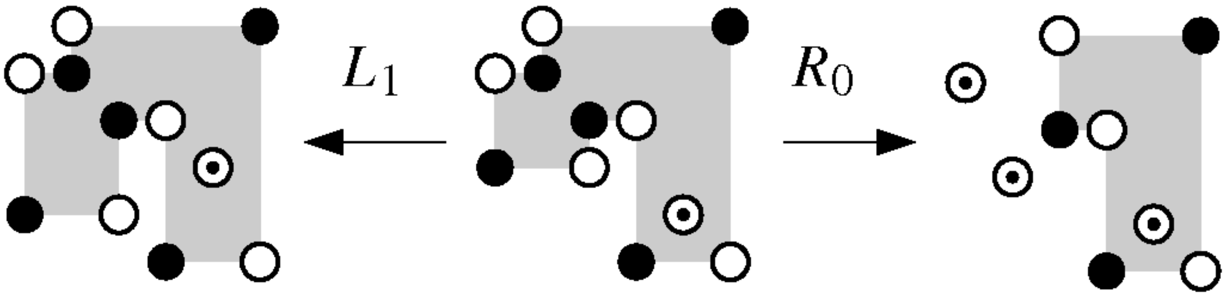


Many are equal

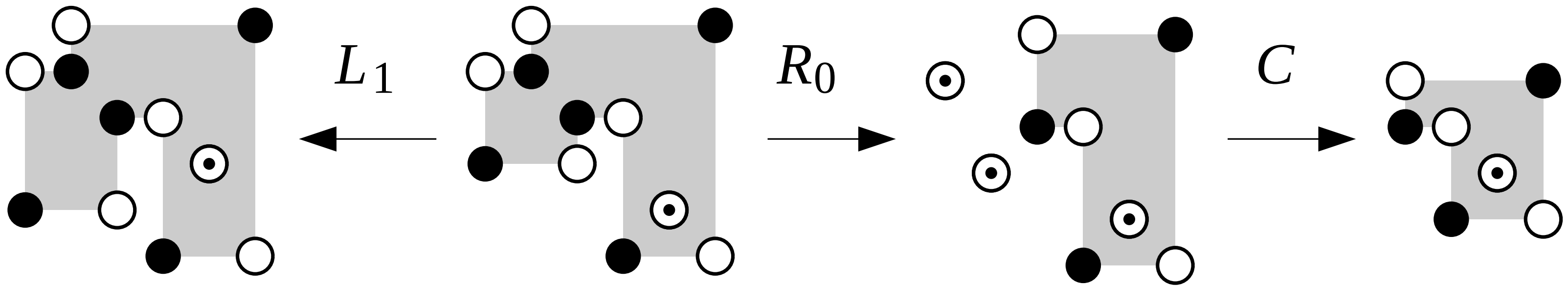


$$\boldsymbol{\mu[\mathcal{L}_i x, \mathcal{L}_i w] = \mu[x, w] = \mu[x\mathcal{R}_i, w\mathcal{R}_i]}$$

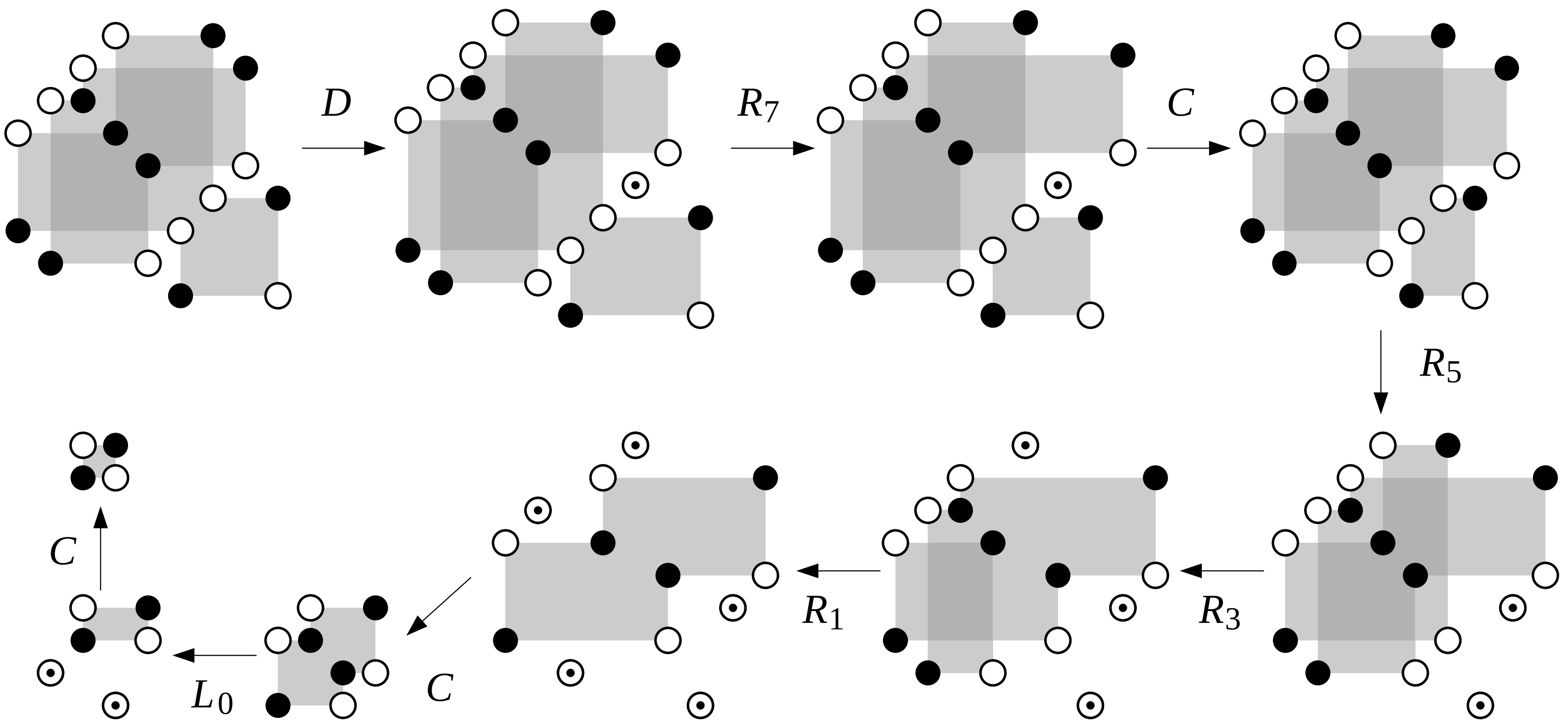
L-S Moves



Compression



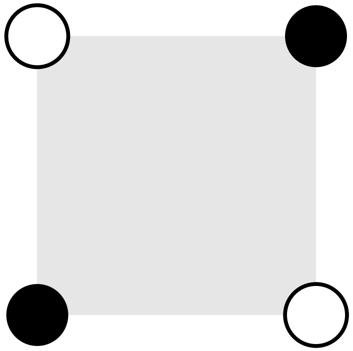
The whole kit and kaboodle



There can be only one

Theorem [W '10]

The only equivalence class that intersects $\boldsymbol{S_2, S_3, \ldots, S_9}$ is that of

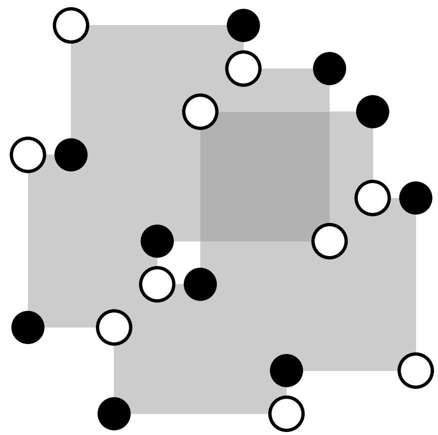


$\boldsymbol{S}_{10}$

Only four additional equivalence classes intersect $\boldsymbol{S}_{10}$.

S_{10}

1



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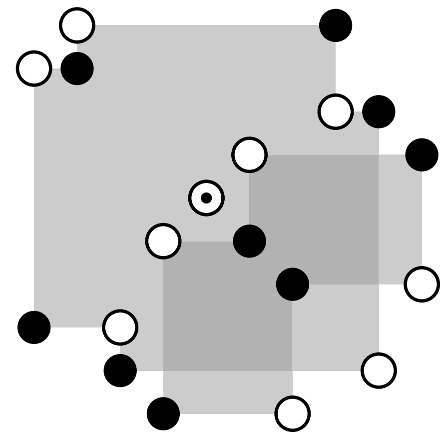

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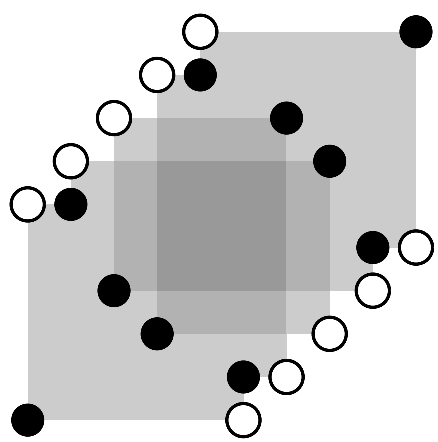

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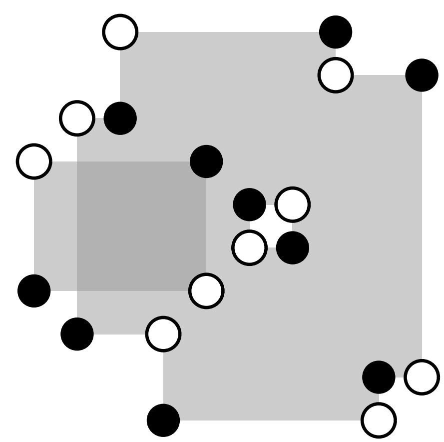
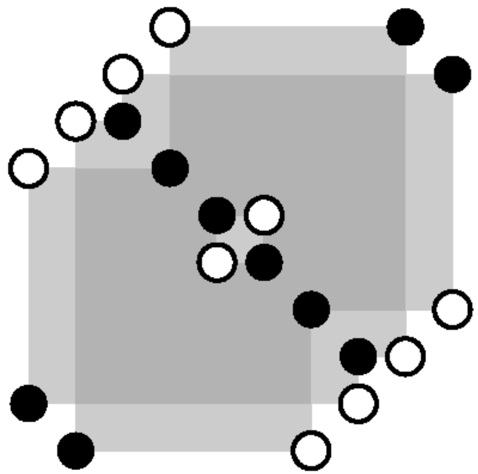


S_{10}

1



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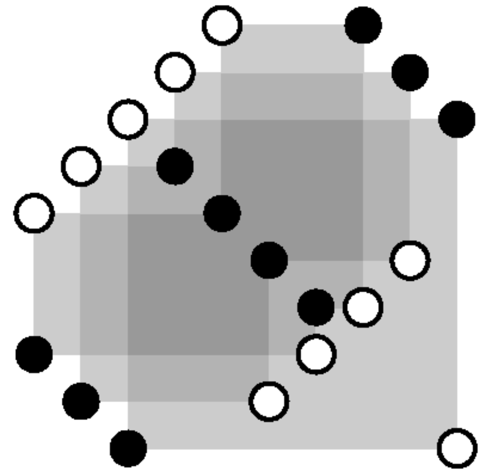

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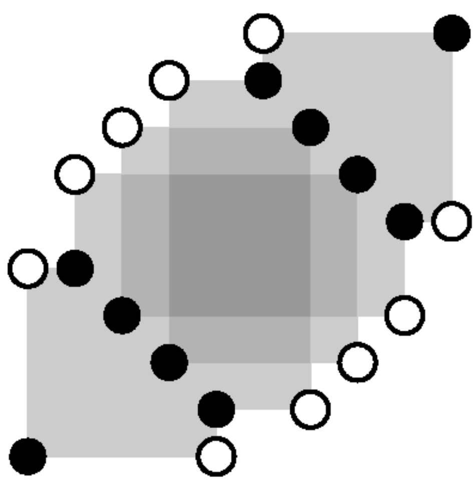

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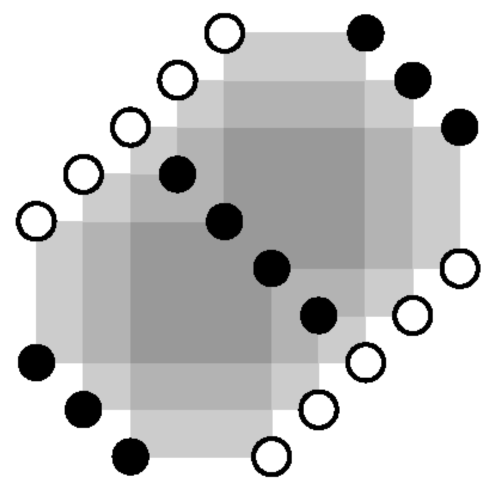
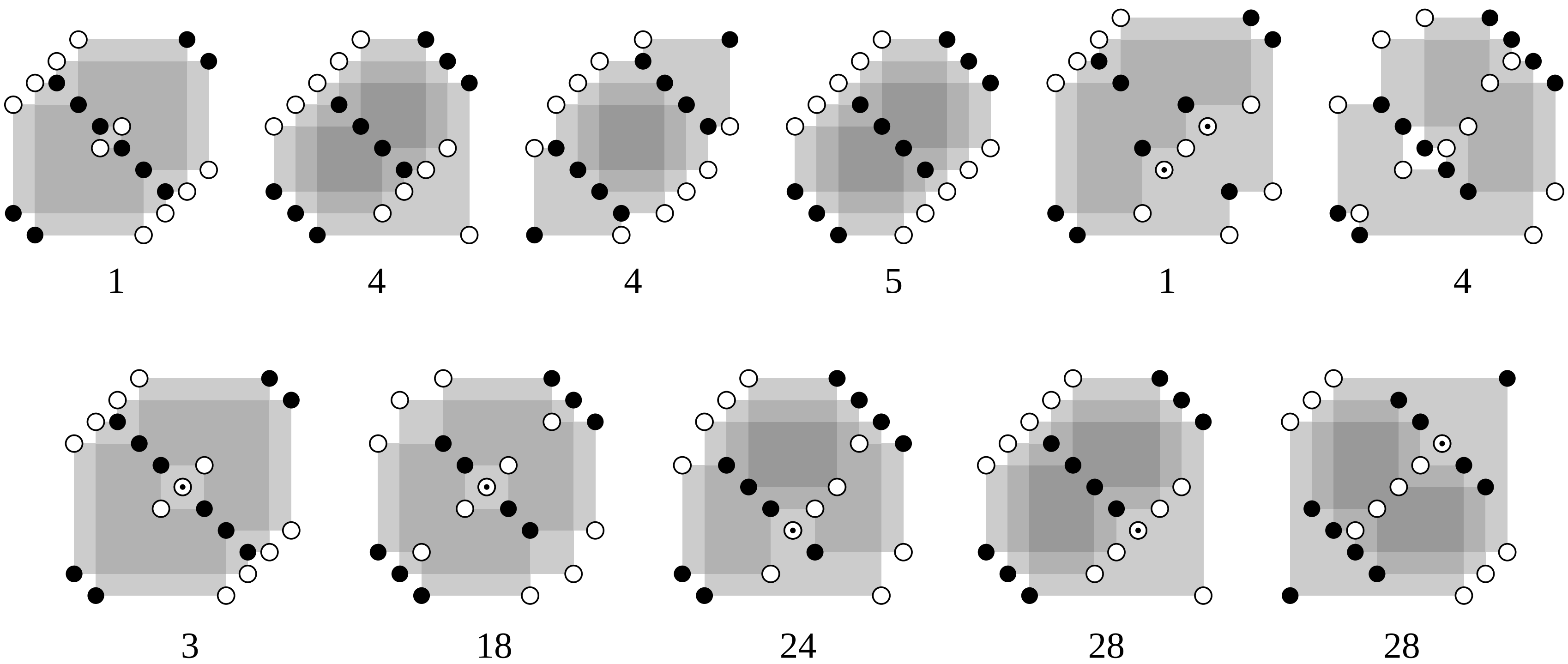
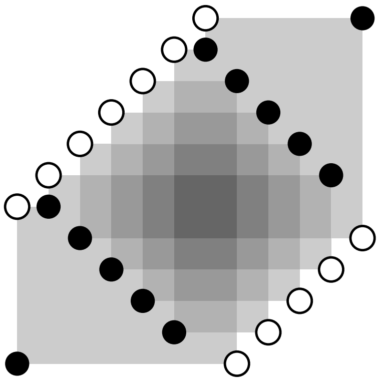


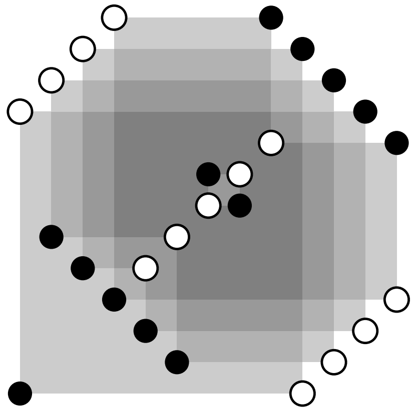
S_{10} and S_{11}



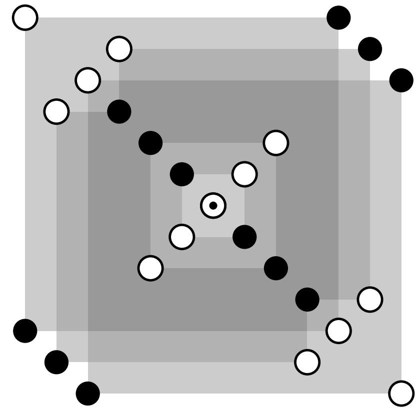
S_{10} and S_{11}



4 (n=10)



1



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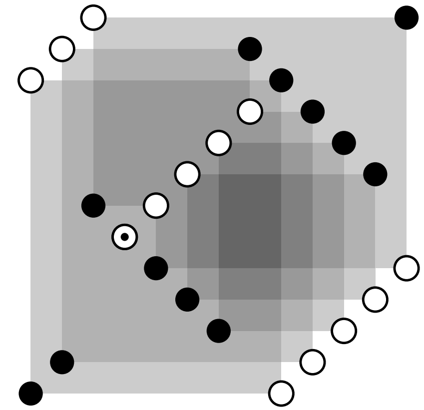

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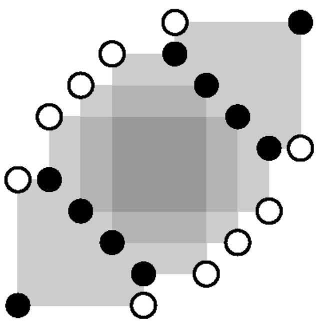

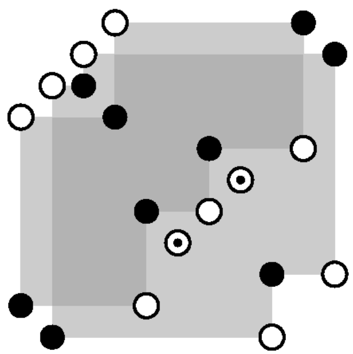

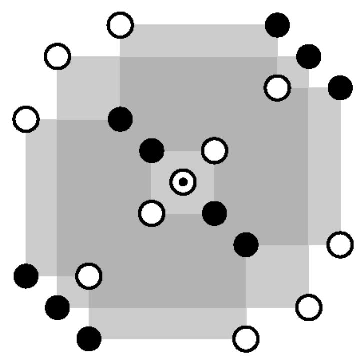

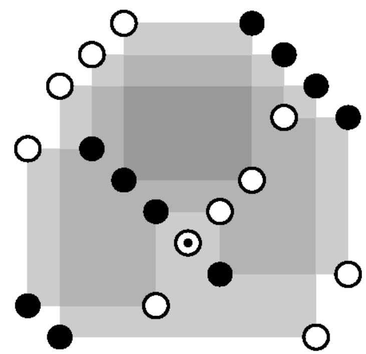

Guess the μ-value