Introduction to Linear Algebra, Spring 2007
MATH 3130, Section 001

EXAM 1

YOUR NAME:

Instructions: Answer each of the following questions completely. To receive full credit, you must *justify* each of your answers (unless stated otherwise). How you reached your answer is more important than the answer itself. If something is unclear, or if you have any questions, then please ask. Good luck!

1. Let A be a 3×3 coefficient matrix corresponding to a system of linear equations that is row equivalent to one of the following matrices in echelon form.

$$M_1 = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \qquad M_2 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

For each of the following statements, state whether A is row equivalent to M_1 or M_2. You should write either M_1 or M_2 in each blank provided. You do *not* need to justify your answers. [**4 points each**]

______ $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

______ The columns of A are linearly dependent.

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______ The columns of A span $\mathbb{R}^3$.

______ The planes corresponding to the 3 linear equations all intersect at a unique point.

______ There exists at least one vector $\mathbf{b} \in \mathbb{R}^3$ such that $A\mathbf{x} = \mathbf{b}$ does *not* have a solution.

2. Consider the following system of linear equations.

$$\begin{array}{rcrcrcr} x_1 & + & 3x_2 & + & x_3 & = & 1 \\ -4x_1 & - & 9x_2 & + & 2x_3 & = & -1 \\ & - & 3x_2 & - & 6x_3 & = & -3 \end{array}$$

(a) Describe the solution set of the system in *parametric vector form.* [**12 points**]

(b) What does the solution set look like geometrically? Be specific. [**4 points**]

3. Consider the following collection of vectors.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -5 \\ 7 \\ 8 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ h \end{bmatrix}$$

(a) Find the values of h for which the vectors are linearly *dependent*. Be sure to justify your answer. **[12 points]**

(b) Find the values of h for which $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \mathbb{R}^3$. Again, be sure to justify your answer. **[12 points]**

4. Assume that $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a *linear* transformation such that

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 5 \end{bmatrix}, T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \text{ and } T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Use this information to find $T\left(\begin{bmatrix} -2 \\ 3 \\ 4 \end{bmatrix}\right)$. **[12 points]**

5. The following statement is *FALSE* in general:

If A is an $m \times n$ matrix such that $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, then $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b} \in \mathbb{R}^m$.

(a) Provide a counterexample where $m = 3$ and $n = 2$ and briefly justify your answer. Your counterexample should consist of a 3×2 matrix A and a vector $\mathbf{b}$. [**12 points**]

(b) The above statement is *TRUE* if $m = n$. Prove this for $m = n = 3$. *Hint:* Think about where the pivots are in the 3×3 coefficient matrix A if $A\mathbf{x} = \mathbf{0}$ has only the trivial solution; looking at problem number 1 might help, as well. [**12 points**]