Introduction to Linear Algebra, Spring 2007
MATH 3130, Section 001

SOLUTIONS TO EXAM 1

YOUR NAME: SOLUTIONS

Instructions: Answer each of the following questions completely. To receive full credit, you must *justify* each of your answers (unless stated otherwise). How you reached your answer is more important than the answer itself. If something is unclear, or if you have any questions, then please ask. Good luck!

1. Let A be a 3×3 coefficient matrix corresponding to a system of linear equations that is row equivalent to one of the following matrices in echelon form.

$$M_1 = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \qquad M_2 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

For each of the following statements, state whether A is row equivalent to M_1 or M_2. You should write either M_1 or M_2 in each blank provided. You do *not* need to justify your answers. [**4 points each**]

$\underline{M_1}$ $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. [**Blue Box on pg 50**]

$\underline{M_2}$ The columns of A are linearly dependent. [**Blue Box on pg 66 and pg 50**]

$\underline{M_1}$ The columns of A are linearly independent. [**Blue Box on pg 66 and pg 50**]

$\underline{M_1}$ The columns of A span $\mathbb{R}^3$. [**Theorem 4**]

$\underline{M_1}$ The planes corresponding to the 3 linear equations all intersect at a unique point. [**Theorem 2**]

$\underline{M_2}$ There exists at least one vector $\mathbf{b} \in \mathbb{R}^3$ such that $A\mathbf{x} = \mathbf{b}$ does *not* have a solution. [**Theorem 4**]

2. Consider the following system of linear equations.

$$\begin{array}{rcrcrcr} x_1 & + & 3x_2 & + & x_3 & = & 1 \\ -4x_1 & - & 9x_2 & + & 2x_3 & = & -1 \\ & - & 3x_2 & - & 6x_3 & = & -3 \end{array}$$

(a) Describe the solution set of the system in *parametric vector form.* [**12 points**]

Solution: First, put the system in a 3×4 augmented matrix. Then row reduce the augmented matrix to reduced echelon form (RREF).

$$\left[\begin{array}{rrr|r} 1 & 3 & 1 & 1 \\ -4 & -9 & 2 & -1 \\ 0 & -3 & -6 & -3 \end{array}\right] \to \cdots \to \left[\begin{array}{rrr|r} 1 & 0 & -5 & -2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

By looking at the reduced matrix, we see that x_3 is free. So, we can write the solution set in parametric vector form in terms of x_3. Let $x_3 = t$ be any real number. Then the solution set is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5t - 2 \\ -2t + 1 \\ t \end{bmatrix} = t \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

(b) What does the solution set look like geometrically? Be specific. [**4 points**]

Solution: Since there is one free variable in the parametric equation above, the solution set is a line. Specifically, it is a line parallel to the vector $\langle 5, -2, 1 \rangle$ and passing through the vector $\langle -2, 1, 0 \rangle$.

3. Consider the following collection of vectors.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -5 \\ 7 \\ 8 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ h \end{bmatrix}$$

(a) Find the values of h for which the vectors are linearly *dependent*. Be sure to justify your answer. **[12 points]**

Solution: Make the vectors the columns of a 3×3 matrix and then row reduce to echelon form (REF).

$$\begin{bmatrix} 1 & -5 & 1 \\ -1 & 7 & 1 \\ -3 & 8 & h \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & -5 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 10+h \end{bmatrix}$$

By looking at the matrix, we see that if $h = -10$, then there will not be a pivot in the third column, which implies that there will be a free variable. In this case, we can find a nontrivial linear combination of $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$ equal to $\mathbf{0}$. Therefore, the three vectors are linearly dependent when $h = -10$.

(b) Find the values of h for which $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \mathbb{R}^3$. Again, be sure to justify your answer. **[12 points]**

Solution: By looking at the reduced matrix above, we see that if $h \neq -10$, then there will be a pivot in every row. By Theorem 4, the columns of the matrix span $\mathbb{R}^3$. That is, $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \mathbb{R}^3$ when $h \neq -10$.

4. Assume that $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a *linear* transformation such that

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 5 \end{bmatrix}, T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \text{ and } T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Use this information to find $T\left(\begin{bmatrix} -2 \\ 3 \\ 4 \end{bmatrix}\right)$. [**12 points**]

Solution: First, note that

$$\begin{bmatrix} -2 \\ 3 \\ 4 \end{bmatrix} = -2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Since T is linear, we can write

$$\begin{aligned} T\left(\begin{bmatrix} -2 \\ 3 \\ 4 \end{bmatrix}\right) &= T\left(-2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) \\ &= -2T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) + 3T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) + 4T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) \\ &= -2\begin{bmatrix} 3 \\ 5 \end{bmatrix} + 3\begin{bmatrix} -1 \\ 2 \end{bmatrix} + 4\begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -9 \\ 0 \end{bmatrix}. \end{aligned}$$

5. The following statement is *FALSE* in general:

   If A is an $m \times n$ matrix such that $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, then $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b} \in \mathbb{R}^m$.

   (a) Provide a counterexample where $m = 3$ and $n = 2$ and briefly justify your answer. Your counterexample should consist of a 3×2 matrix A and a vector $\mathbf{b}$. [**12 points**]

   Counterexample: Any example that results in 3×3 augmented matrix that is inconsistent will do. How about these? Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

   Since A has a pivot in every column, there are no free variables, which implies that $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. However, the following augmented matrix is equivalent to $A\mathbf{x} = \mathbf{b}$.

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

   By looking at this matrix, we see that there is a pivot in the augmented column. By Theorem 2, $A\mathbf{x} = \mathbf{b}$ is inconsistent. That is, $A\mathbf{x} = \mathbf{b}$ does not have a solution for the above A and $\mathbf{b}$.

   (b) The above statement is *TRUE* if $m = n$. Prove this for $m = n = 3$. *Hint:* Think about where the pivots are in the 3×3 coefficient matrix A if $A\mathbf{x} = \mathbf{0}$ has only the trivial solution; looking at problem number 1 might help, as well. [**12 points**]

   Proof: Let A be a 3×3 matrix such that $A\mathbf{x} = \mathbf{b}$ has only the trivial solution. Then A has no free variables, which implies that A has a pivot in every *column*. But A has the same number of columns as rows. So, A has a pivot in every *row*. By Theorem 4, $A\mathbf{x} = \mathbf{b}$ has a solution for all $\mathbf{b} \in \mathbb{R}^3$.

   $\square$