Introduction to Linear Algebra, Spring 2007
MATH 3130, Section 001
SOLUTIONS TO PROJECT 2

1. Prove Theorem 2.3(d).

   Proof: Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be $n \times n$ matrices. We will show that the (i, j)-entry of $(AB)^T$ equals the (i, j)-entry of $B^T A^T$ for arbitrary $1 \leq i, j \leq n$. The (i, j)-entry of $(AB)^T$ is the (j, i)-entry of AB (by definition). The (j, i)-entry of AB is the dot product of the jth row of A with the ith column of B (by definition):

   $$(a_{j1}, \ldots, a_{jn}) \cdot (b_{1i}, \ldots, b_{ni}) = a_{j1}b_{1i} + \cdots + a_{jn}b_{ni}.$$

   On the other hand, the (i, j)-entry of $B^T A^T$ is the dot product of the ith row of B^T with the jth column of B^T. But the ith row of B^T is the ith column of B, and the jth column of A^T is the jth row of A. So, the (i, j)-entry of $B^T A^T$ is

   $$(b_{1i}, \ldots, b_{ni}) \cdot (a_{j1}, \ldots, a_{jn}) = a_{j1}b_{1i} + \cdots + a_{jn}b_{ni},$$

   which is the same as above. Therefore, $(AB)^T = B^T A^T$.

2. Let V be a vector space (over $\mathbb{R}$). Let H and K be subspaces of V. The intersection of H and K, written as $H \cap K$, is the set of all $\mathbf{v} \in V$ that belong to both H and K. Prove that $H \cap K$ is a subspace of V.

   Proof: We have three things to check.

   (i) Since H and K are subspaces, $\mathbf{0} \in H$ and $\mathbf{0} \in K$. Therefore, $\mathbf{0} \in H \cap K$.

   (ii) Let $\mathbf{u}, \mathbf{v} \in H \cap K$. Then $\mathbf{u}, \mathbf{v} \in H$ and $\mathbf{u}, \mathbf{v} \in K$. Since H and K are subspaces, $\mathbf{u} + \mathbf{v} \in H$ and $\mathbf{u} + \mathbf{v} \in K$. Therefore, $\mathbf{u} + \mathbf{v} \in H \cap K$.

   (iii) Let $\mathbf{u} \in H \cap K$ and $c \in \mathbb{R}$. Then $\mathbf{u} \in H$ and $\mathbf{u} \in K$. Since H and K are subspaces, $c\mathbf{u} \in H$ and $c\mathbf{u} \in K$. Therefore, $c\mathbf{u} \in H \cap K$.

   Since (i), (ii), and (iii) hold, $H \cap K$ is a subspace of V.

3. Let $V = \mathbb{R}^2$ with the usual scalar multiplication, but instead of the usual definition of addition, define

   $$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2 + 1, y_1 + y_2 + 1).$$

   Determine whether V is a vector space. If V is a vector space, then prove it. If V is not a vector space, then show that one of the axioms of vector spaces fails.

   Solution: V is *not* a vector space. At first glance, you might think that V is not a vector space because it doesn't appear to have a vector that acts like the zero vector. But in fact, $(-1, -1)$ plays the role of the zero vector. One of the axioms of the definition of vector space that does fail is Axiom 7. Let $c = d = 1$ and let $\mathbf{u} = (1, 1)$. Then

   $$(c + d)\mathbf{u} = (1 + 1)(1, 1) = 2(1, 1) = (2, 2).$$

On the other hand,

$$c\mathbf{u} + d\mathbf{u} = 1(1,1) + 1(1,1) = (1,1) + (1,1) = (2+1, 2+1) = (3,3).$$

Since $(2,2) \neq (3,3)$, Axiom 7 fails. Therefore, V is not a vector space.

4. Read Section 1.10 and complete Exercise 10 on page 101.

   Solution: For this problem, the migration matrix is

$$M = \begin{bmatrix} .93 & .03 \\ .07 & .97 \end{bmatrix}.$$

In 2000, we are given that

$$\mathbf{x}_0 = \begin{bmatrix} 800,000 \\ 500,000 \end{bmatrix}.$$

Then in 2002, we have

$$\mathbf{x}_2 = M^2\mathbf{x}_0 = \begin{bmatrix} .93 & .03 \\ .07 & .97 \end{bmatrix}\begin{bmatrix} .93 & .03 \\ .07 & .97 \end{bmatrix}\begin{bmatrix} 800,000 \\ 500,000 \end{bmatrix} = \begin{bmatrix} 722,100 \\ 577,900 \end{bmatrix}.$$

Therefore, there will be 722,100 people in the city and 577,900 people in the suburbs in 2002.

5. Read Section 2.7 and complete Exercise 18 on page 166.

   Solution: Let T_1 be the (linear) transformation that rotates points in $\mathbb{R}^3$ about the z-axis through an angle of $-30°$. Then $T_1(\mathbf{e}_1) = (\cos -30°, \sin -30°)$, $T_1(\mathbf{e}_2) = (\cos 60°, \sin 60°)$, and $T_1(\mathbf{e}_3) = \mathbf{e}_3$. Then the matrix for this rotation for homogeneous coordinates is

$$\begin{bmatrix} \sqrt{3}/2 & 1/2 & 0 & 0 \\ -1/2 & \sqrt{3}/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

For the translation, we want $(x, y, z, 1)$ to map to $(x+5, y-2, z+1, 1)$. The matrix that does this is

$$\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Therefore, the matrix that first rotates and then translates is

$$\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \sqrt{3}/2 & 1/2 & 0 & 0 \\ -1/2 & \sqrt{3}/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & 1/2 & 0 & 5 \\ -1/2 & \sqrt{3}/2 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$