Introduction to Linear Algebra, Spring 2007
MATH 3130, Section 001

PROJECT 1

NAME 1:
NAME 2:
NAME 3:

Instructions: In groups of 2 to 3 people (preferably 3), answer any 3 of the following questions. If you complete more than 3 questions, I will grade the first 3. Each group is required to turn in one set of solutions. Your answers should be written up on separate paper and *neatly presented* (do not write your answers on this sheet). In order to receive full credit on each question, you must justify your answers and show all of your work. When you are using a theorem (or fact in a Blue Box) to make a conclusion, be sure to state which theorem you are using and where you are using it. Use this document as the cover page to your project. The names of the people in your group should appear above. Also, circle the problems below that you have completed. Each problem is worth 10 points. This project is due Friday, February 23. Please come talk to me if you have questions. Enjoy!

1. Consider a system of m linear equations each in n variables.

   (a) Prove that the 3 row operations preserve the solution set of the linear system. *Note:* This proof is not difficult if you start off correctly. If you need help getting started, ask for some guidance. The discussion of this fact in Section 1.1 of our text book is not rigorous enough, so don't copy that. [**7 points**]

   (b) What geometric effect does each row operation have on the hyperplanes that correspond to the linear equations in the system? Be as specific as possible and *briefly* justify your answer. [**3 points**]

2. If $k \geq 1$, then a *k-dimensional linear subspace* of $\mathbb{R}^n$ is equal to the span of some collection of k linearly independent vectors from $\mathbb{R}^n$. Geometrically, these are lines (1-dimensional), planes (2-dimensional), etc. passing through the origin in $\mathbb{R}^n$. The *0-dimensional linear subspace* of $\mathbb{R}^n$ is the zero-vector, $\mathbf{0}$. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ be a collection of linearly independent vectors from $\mathbb{R}^n$ and let $W = \text{span}\{\mathbf{v}_1, \ldots \mathbf{v}_k\}$, so that W is a k-dimensional linear subspace of $\mathbb{R}^n$. Note that $1 \leq k \leq n$. Now, let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Define $T(W) = \{T(\mathbf{x}) : \mathbf{x} \in W\}$.

   (a) Prove that $T(W)$ is a linear subspace of $\mathbb{R}^m$. [**7 points**]

   (b) What are the possible dimensions of $T(W)$? [**3 points**].

3. Read Section 1.6 and complete Exercise 12. [**10 points**]

4. Let $T_1 : \mathbb{R}^n \to \mathbb{R}^m$ and $T_2 : \mathbb{R}^m \to \mathbb{R}^k$ be linear transformations. Define $S : \mathbb{R}^n \to \mathbb{R}^k$ via $S(\mathbf{x}) = T_2\left(T_1(\mathbf{x})\right)$.

   (a) Prove that S is a linear transformation. [**7 points**]

   (b) When will S be one-to-one and onto? Justify your answer. [**3 points**]

5. Suppose $AB = I_n$ (the $n \times n$ identity matrix).

   (a) Prove that $A\mathbf{x} = \mathbf{b}$ is consistent for all $b \in \mathbb{R}^n$. [**7 points**]

   (b) Prove that A must have at least as many columns as rows. [**3 points**]