Fixed Points and Minimax Theorems

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14 Nov. 06

Backround

Definition 1: Suppose X, Y are nonempty sets and let $f : X \times Y \to \mathbb{R}$. We say f satisfies the minimax condition if

$$\sup_{y \in Y} \inf_{x \in X} f(x, y) = \inf_{x \in X} \sup_{y \in Y} f(x, y).$$

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Note: The theorem is still true if we replace D^n with any space homeomorphic to D^n.

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1952,1959: Fan Publishes his results concerning minimax theorems.

Proposition 1: If X, Y are nonempty and if $f : X \times Y \to \mathbb{R}$, then

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Definition 2: Let X, Y be two nonempty sets. Let$f$ be a real-valued function defined on $X \times Y$. We say $(\bar{x}, \bar{y}) \in X \times Y$ is a saddle point if

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Proposition 2: Suppose X, Y are nonempty sets and let f be a real valued function on $X \times Y$. Then $(\bar{x}, \bar{y})$ is a saddle point if and only if

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Proof: More Inequality-ology

Main Lemma (Fan 1959): Let X be a nonempty compact convex subset of a topological vector space. Let f be a continuous real-valued function on $X \times X$. Suppose that for each fixed $y \in X$, $f(x, y)$ is convex function of x. Then there exists $y_o \in X$ such that $f(x, y_o) \geq f(y_o, y_o)$ for all $x \in X$.

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Let $\{x_1, ..., x_n\} \subseteq X$ and let $K = co(x_1, ..., x_n)$ be their convex hull. Since X is convex, $K \subseteq X$.

For each $i = 1, 2, ..., n$, define $g_i : K \to \mathbb{R}$ via $g_i(y) = \max\{f(y, y) - f(x_i, y), 0\}$, for $y \in K$. Then each g_i is continuous and non-negative.

For any $y \in K$, write $y = \sum_{i=1}^{n} a_i x_i$, where $a_i \geq 0$ and $\sum_{i=1}^{n} a_i = 1$.

Then TFAE:

a) $f(x_i, y) \geq f(y, y)$, for $i = 1, 2, ..., n$

b) $g_i(y) = 0$, for $i = 1, 2, ..., n$

c) $a_i \sum_{k=1}^{n} g_k(y) = g_i(y)$, for $i = 1, 2, ..., n$

4.5

Next, let $S = \{(a_1, ..., a_n) \in \mathbb{R}^n | a_i \geq 0, \sum a_i = 1\}$ Then S is a compact subset of $\mathbb{R}^n$. Let $\psi : S \to S$ via $\psi(a_1, ..., a_n) = (a'_1, ..., a'_n)$, where

$$a'_i = \frac{a_i + g_i(\sum_{j=1}^n a_j x_j)}{1 + \sum_{k=1}^n g_k(\sum_{j=1}^n a_j x_j)}$$

Note that ψ really doe map into S. Also, ψ is continuous since each component function is continuous. Since ψ is a continuous map from the compact S into itself,the Brouwer Fixed Point Theorem says ψ must have some fixed point, $(a_1, ..., a_n)$. Let $y = \sum_{i=1}^n a_i x_i$. Then for each $i = 1, ...n$

$$a_i = \frac{a_i + g_i(y)}{1 + \sum_{k=1}^n g_k(y)}$$

$$\Rightarrow a_i \sum_{k=1}^n g_k(y) = g_i(y)$$

4.7

But this is condition (c) from above, which is equivalent to

$$f(x_i, y) \geq f(y, y), \text{for} i = 1, 2...., n.$$

Thus $y \in \bigcap_{i=1}^{n} F_{x_i} \Rightarrow \bigcap_{i=1}^{n} F_{x_i} \neq \emptyset \Rightarrow \bigcap_{x \in X} F_x \neq \emptyset$. ■

Lemma 2: Suppose L_1, L_2 are topological vector spaces and suppose $X_1 \subseteq L_1, X_2 \subseteq L_2$ are nonempty compact convex subsets. Let $f_1, f_2 : X = X_1 \times X_2 \to \mathbb{R}$ be continuous functions such that $f_1(x, y)$ is concave in x for each fixed y and $f_2(x, y)$ is concave in y for each fixed x. Then there exists $\hat{x} = (\hat{x}_1, \hat{x}_2) \in X$ such that

$$f_1(\hat{x}_1, \hat{x}_2) = \max_{x_1 \in X_1} f(x_1, \hat{x}_2) \quad \text{and}$$

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Proof: Apply the previous lemma to $g : X \times X \to \mathbb{R}$ given by

$$g(x, y) = f_1(x_1, y_2) + f_2(y_1, x_2), \text{where } x = (x_1, x_2); y = (y_1, y_2).$$

This gives an $\hat{x} = (\hat{x_1}, \hat{x_2})$. Check that this works.

5.5

Theorem(Fan 1959): Let X, Y be nonempty compact convex sets, each in a topological vector space. Suppose that f is real-valued continuous function on $X \times Y$ such that for each fixed $y \in Y$, $f(x, y)$ is a convex function of x, and for each fixed $x \in X$, $f(x, y)$ is a concave function of y. Then there exists $\bar{x} \in X$ and $\bar{y} \in Y$ such that $(\bar{x}, \bar{y})$ is a saddle point of f, i.e.

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Proof: Apply lemma 2 to $f_1 = -f$ and $f_2 = f$. Check it all works.

Theorem (Von Neumann): Let c_{ik} be an arbitrary set of real numbers, $1 \leq i \leq n, 1 \leq k \leq m$. Define the sets

$$S = \{\vec{\xi} = (\xi_1, \xi_2, ..., \xi_n) \in \mathbb{R}^n | \xi_i \geq 0, \sum_{i=1}^{n} \xi_i = 1\}$$

$$T = \{\vec{\eta} = (\eta_1, \eta_2, ..., \eta_m) \in \mathbb{R}^m | \eta_k \geq 0, \sum_{k=1}^{m} \eta_k = 1\}$$

Define $K : S \times T \to \mathbb{R}$ via $K(\vec{\xi}, \vec{\eta}) = \sum_{i=1}^{n} \sum_{k=1}^{m} c_{ik} \xi_i \eta_k$. Then there exists $(\vec{\xi}_o, \vec{\eta}_o) \in S \times T$ such that

$$K(\vec{\xi}_o, \vec{\eta}_o) = \min_{\vec{\xi} \in S} \max_{\vec{\eta} \in T} K(\vec{\xi}, \vec{\eta}) = \max_{\vec{\eta} \in T} \min_{\vec{\xi} \in S} K(\vec{\xi}, \vec{\eta})$$

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Proof: Use Fan's Theorem: S & T are compact convex subsets and K is linear in $\vec{\xi}$ for fixed $\vec{\eta}$ (and thus convex in $\vec{\xi}$ for fixed $\vec{\eta}$) and linear in $\vec{\eta}$ for fixed $\vec{\xi}$ (and thus concave in $\vec{\eta}$ for fixed $\vec{\xi}$). ■

Lemma 3(Fan,1952): Let X, Y be nonempty compact convex sets, each in a topological vector space. Suppose that f is a real-valued continuous function on $X \times Y$ such that for each fixed $y \in Y$, $f(x, y)$ is a lower semicontinuous function of x, and for each fixed $x \in X$, $f(x, y)$ is an upper semicontinuous function of y. Then

$$\min_{x \in X} \max_{y \in Y} f(x, y) = \max_{y \in Y} \min_{x \in X} f(x, y)$$

holds if and only if the following condition is satisfied: For any two finite sets $\{x_1, x_2, ..., x_n\} \subseteq X$ and $\{y_1, y_2, ..., y_m\} \subseteq Y$, there exists $(x_o, y_o) \in X \times Y$ such that

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Proof: Similar to the proof of the Main Lemma.

Theorem(Fan 1959): Let X, Y be nonempty compact convex sets, each in a topological vector space and let f be a real-valued function on $X \times Y$. Suppose that for each fixed $y \in Y$, $f(x, y)$ is a continuous convex function of x, and for each fixed $x \in X$, $f(x, y)$ is a continuous concave function of y. Then there exists $\bar{x} \in X$ and $\bar{y} \in Y$ such that $(\bar{x}, \bar{y})$ is a saddle point of f, i.e.

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Proof: Use lemma 3 and Von Neumann's Theorem.

Bibliography

[1] Lorin W. Woo, Maximal Points of Convex Sets in Locally Convex Topological Vector Spaces, 1999

[2] Ky Fan, Minimax Theorems, *Proc. Nat. Acad. Sci.* 39:42-47

[3] John Nash, Equilibrium Points in N-Person Games, *Proc. Nat. Acad. Sci.*36:48-49

[4] James R. Munkres, *Topology, 2nd Edition*, Prentice Hall Inc.,1975