Good Problems: teaching mathematical writing

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It is important for scientists and engineers to be able to communicate mathematical ideas in writing. Science and engineering students, however, generally do not have this ability and believe that it is not important. Little effort is spent, however, on *teaching* the students how to write, or that writing is important.

We present here a plan to make writing coherent mathematics a regular part of the students' experience. This will both improve their writing and teach them that writing is valued in mathematics. We provide materials to teach a set of important writing skills, so the students will not have to guess how to write. This method is designed to be as painless as possible, for the instructors as well as for the students.

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The Good Problems Method

The main goal of this method is to teach mathematical writing. In many universities students are expected to be able to write technology-based laboratory reports for their sophomore level courses. For many of them (it seems) this is the first time they have been expected to write a coherent report on a mathematical subject. In any case, many have not mastered a basic set of writing skills. There is a perception among the students that engineers and scientists do not need to know how to write. For most college courses this is true, but we know that after college, it is crucial to be able to write about what you do.

The method itself is very simple. Each week, in conjunction with the regular homework assignment, one homework problem will be designated a "good problem." The student must write the solution to this problem carefully, demonstrating specific skills at presenting mathematics. A set of six skills will be taught during the course of a semester. At any given point in the semester, the student will be graded only on those skills that have already been taught.

The main mechanism for teaching these skills is a set of handouts. Each handout identifies and explains a specific skill, and provides examples of good use of this skill and examples where this skill is lacking. The examples are taken from differential Calculus, so the handouts can be used at that level or above. A secondary mechanism for teaching these skills can be discussions or demonstrations during the recitation. After a skill is officially learned, the regular feedback from these weekly assignments will coax the students toward mastery of that skill.

One beneficial side effect of this program is to encourage organization. College presents newfound pressures on the students' time and energy. It is expected that to survive the student will have to learn time-management and organizational skills. This topic is beyond the scope of our endeavor, but we require the student to present one well-organized sheet of paper each week.

A second beneficial side effect is to encourage logical thinking. This is another skill that is vital, but not taught directly. By teaching the students the proper uses of logical connectives, we can teach them to recognize when their argument has gaps or contradictions. It is again beyond our scope to teach logical thinking properly, but we can require one logical sheet of paper each week, and provide some basic skills.

One very important aspect of this method is that it is designed to be as painless as possible. One drawback of many teaching reforms is that they require significantly more effort on the part of both the instructors and the students. When performance gains are noted, it is not clear how much is simply due to spending more time thinking about mathematics. We aim here not to add effort to the system, but to re-align a small amount of effort to a more productive area.

This packet contains all the materials required to implement this method. There is some initial added labor involved of course, but then instructors need merely assemble the ingredients provided and set the program in motion. The teaching assistants will need to spend time brushing up on their own writing skills, but this will benefit them, and so should not be considered a burden. In the short term, we are requiring the students to learn additional skills and they will therefore have to expend additional effort. We expect, however, the returns to be rather quick. Many of these 'presentation' skills transfer directly to problem-solving skills, and also help with reading mathematics. We expect these skills to carry over to future mathematics courses.

How to Use This Packet

This packet contains all the materials necessary to implement the good problems program. The items included are:

The Instructor's Guide This guide is for the main instructor, meaning the person in charge of organizing the course. It lists those tasks that the instructor must perform to implement this scheme, indicates when they must be performed, and estimates how much time they are expected to take. If there is more than one instructor, then several of these tasks can be shared.

The Teaching Assistant's Guide This guide is for the teaching assistant, who runs the recitations. It lists those tasks that the teaching assistant must perform, indicates when they must be performed, and estimates the time required. If the course is taught without recitations, these tasks fall on the instructor.

The Student's Guide This guide explains the method to the students, and tells them what is expected from them. When this program is implemented, this guide could be one side of a sheet of paper and the schedule of good problems for the course photocopied onto the back.

The Handouts The main material for this method is a set of six handouts. Each handout is two pages long, so that it can be copied double-sided onto a single sheet of paper. The handout consists of a description of a particular writing skill, motivation for the importance of this skill, whatever explicit rules can be given, and examples of good and bad presentation emphasizing that skill. It is hoped that the students will find these handouts useful enough that they will keep them for future reference.

The handouts can be used in any order. We suggest the following ordering:

1. Laying out the Problem.
2. Flow.
3. Mathematical Symbols.
4. Logical Connectives.
5. Graphs.
6. Introductions and Conclusions.

Instructor's Guide to Good Problems

This guide is for the main instructor, meaning the person in charge of organizing the course. It lists those tasks that the instructor must perform to implement this scheme, indicates when they must be performed, and estimates how much time they are expected to take. If there is more than one instructor, then several of these tasks can be shared.

The first time you use this method, there are extra tasks, which we list first.

1. Read this packet, learn this material, make modifications, and decide it is a good thing to do. This should be performed a few months before the semester starts to allow the ideas to settle. Time required: 3 hours.

2. Convince the other people involved with the course to go along with this plan. This task should be completed a month before the semester starts. Time required: 1 hour.

3. Deal with unanticipated problems. Time required: 2 hours.

Each time the method is used, there are the following tasks.

1. Choose the good problems, schedule them, and decide on the time schedule for each new skill. The problems should not be harder than the regular homework problems, but should be substantive enough that there is something to write about. They must also be chosen to be relevant to the new skill learned that week. (If the handout is on graphing, the problem must require a graph.) There are fewer handouts than weeks in the semester, so there is flexibility in scheduling certain skills for certain sections, or avoiding busy (test) weeks. You might consider making the good problem on the previous week's material, since a student will not be able to write about new material that they do not yet understand. This task should be performed when the regular homework problems are chosen. Time required: 1 hour.

2. Provide initial training for teaching assistants. You must introduce this method, distribute materials and discuss its implementation with the teaching assistants. This should occur one week before the semester starts. Time required: 2 hours.

3. Explain the method to the students. The concept should be introduced and the student's guide distributed in the first week of classes. The skills handouts could be distributed all at once at the beginning or as they are used. Time required: 10 minutes of class time.

4. Provide feedback on the handouts. Any deficiencies in the method or the handouts should be noted. At the end of the semester comments from the instructors, teaching assistants, and students should be collected and the materials corrected. Time required: 1 hour.

Remark: Grade the good problem either as a significant portion of the homework or as a quiz. A balance must be struck so the students will put effort into the good problem without neglecting the others.

TA's Guide to Good Problems

This guide is for the teaching assistant, who runs the recitations. It lists those tasks that the teaching assistant must perform, indicates when they must be performed, and estimates the time required. If the course is taught without recitations, these tasks fall on the instructor.

The first time you use this method, there are extra tasks, which we list first.

1. Read and understand the material in this packet. This should be performed in the week before the semester starts. Time required: 2 hours.

2. Attend training. The instructor should provide some training during the week before the semester starts. Time required: 1 hour.

3. Deal with unanticipated problems. Time required: 2 hours.

Each time the method is used there are the following tasks.

1. Prepare for the new skill to be taught that week. It may also be useful to review the old skills. This is part of the weekly preparation time. Time required: 5 minutes per week.

2. Present difficult skills in the recitation. Sometimes students will have particular difficulty on some specific skill. The teaching assistant will need to re-explain this skill and perhaps demonstrate it on a relevant problem. It is important to note that the teaching assistant is not expected to be the main teacher of these skills. The handouts should provide all the information the students need. Time required: 5 minutes recitation time per week, on average.

3. Grade the good problems. It seems fair to grade two less regular homework problems to compensate for the good problems. After some initial trauma, it is expected that the good problems will be more pleasant to grade than ordinary problems. There should be no net gain in grading time as a result of this.

4. Provide feedback on the handouts. Any deficiencies in the method or the handouts should be noted. At the end of the semester these comments should be collected by the instructor and the materials corrected. Time required: 1 hour.

Student's Guide to Good Problems

Many science and engineering students believe that writing has nothing to do with mathematics. **This is false.** In any field, it is important to be able to communicate your ideas and results to others. Exactly how you communicate them varies from field to field, but in nearly all fields it is through writing. Mathematical writing, however, has its own particular style. The emphasis is on clarity and precision, and not on the clever turn of phrase.

The "Good Problems" program is designed to teach you how to write about mathematics coherently. It provides you a set of specific writing skills and gives you enough practice through assigned problems each week so that writing well will become a habit. It is also designed to be as painless as possible.

Each week one homework problem is designated a "good problem". You need to write the solution to this problem carefully, in good "presentation" format. It will be graded partly for correctness, but mainly on presentation.

During the semester you will receive six handouts, each addressing a particular writing skill. The good problem each week is graded only on those skills already covered, with greater emphasis on the newest skill. By the end of the semester you will have acquired the basic skills of mathematical writing. The handouts are:

- Laying out the Problem.
- Flow
- Mathematical Symbols.
- Logical Connectives.
- Graphs.
- Introductions and Conclusions.

Each handout consists of a description of a particular writing skill, motivation for the importance of this skill, whatever explicit rules can be given, and examples of good and bad presentation emphasizing that skill. We use a consistent format for the examples.

Bad: *An example of poor writing looks like this.*

Good: An example of good writing looks like this.

A comment within an example looks like this.

If you already have these basic writing skills, it should take you no more than five extra minutes to write up a good problem. If you do not have these skills, you will have to spend extra effort in the short term to acquire them. In the long term, however, these skills will save you much time and effort. Besides making future writing assignments easier, these skills will help you with problem-solving (by encouraging organization and logic) and with reading mathematics. These skills should be carried into future mathematics courses.

Laying out the Problem

Presentation is very important in mathematics, just as it is in other fields. The main reason for this is clarity. Good presentation allows you to communicate more clearly the content of your work. A secondary reason is the appearance of competence. Careful presentation makes the reader think you were also careful with the content of your work. Although good presentation is rarely successful at bluffing past poor content, poor presentation can easily ruin good content.

The first thing the reader evaluates is the overall visual layout of the problem. Exactly how polished this should be will depend on the purpose of your writing. For our purposes, we have the following rules:

- Put the problem on its own sheet of paper. Regular quality paper is fine, but there should be no ragged edges or tears.
- If you need more than one page, staple (no folded corners!) them together and put your name and the page number on each page.
- Leave margins on all sides.
- Print neatly or type. Do not switch colors or from pen to pencil in the middle of the problem. (You can use different colors to highlight if you wish.)
- If you had to cross out material or erased a lot and left smudges, rewrite the problem. (It is a good habit to solve the problem first on scrap paper and then copy it neatly.)

The reader next needs to know what it is they are reading. On top of the first page, put:

1. your full name,
2. student ID number,
3. the course and recitation number, and
4. the assignment number and/or the date the assignment is due.

Next is the format for the problem itself:

- Label the problem with the chapter, section, and problem number.
- Write out the entire question, including any instructions. If the question refers to another problem, include the relevant information from that problem. The goal is to make your work as self-contained as possible, so the reader does not need to look anything up.
- Do the problem in some logical order. Do not do the problem in several disjoint pieces connected by arrows.

All these rules may seem picky. Once you have learned this way to lay out your work, you will understand the principles behind these rules. Then you will know when to change the rules.

Good presentation is important in mathematics. It lets you communicate your work more clearly and lends it credibility.

Example:

Good:

Full Name
Student ID number
Course and recitation number
Assignment number
Date

Section number, problem number. Full text of the problem . . . blah . . . blah . . . blah . . . blah.

In this problem we . . . blah . . . blah . . . blah . . . blah . . . blah . . . blah . . . blah . . . blah.

There is a separate handout on introductions.

Since . . . blah . . . blah . . . we know . . . blah . . . blah.

There is a separate handout on logical connectives.

$\Longrightarrow$ blah
$\Longrightarrow$ blah blah
$\therefore$ blah!

There is a separate handout on mathematical symbols.

From this we can conclude . . . blah . . . blah . . . blah . . . blah and so $x = 3\pi$ . . . . . . blah.

There is a separate handout on incorporating mathematics into sentences and other issues of 'flow'.

By graphing our solution, we can see that . . . blah.

There is a separate handout on graphing.

By using . . . blah . . . blah . . . we were able to solve this problem . . . blah . . . blah . . . blah . . . blah.

There is a separate handout on conclusions.

Flow

Written mathematics must be readable. This may seem trivial, but it is an important point. You should be able to read your work aloud to a classmate and have them understand your solution. If you need to add any explanations, these should be included in your written work. The most common mistake is to write mathematics without using enough words. All writing, even mathematics, should consist of complete sentences. These should explain the problem by providing both the method and justification for each step of the solution.

Why are sentences important in mathematics?

Although sometimes it seems hard to read textbooks, it would be much harder to understand if they only had equations and no sentences. The situation is similar in lecture; if the professor just listed formulas on the chalkboard without talking about them, leaving you to figure out what was being done in each step, how much could you understand from the lecture? Neither of these would be a good way for most students to learn, since sentences are necessary to explain the mathematics.

How should students use sentences in a Mathematics class?

In a Mathematics class, you should explain homework solutions using complete sentences. That means linking together thoughts with words and embedding equations into sentences. Going through the extra work to do this will benefit you in several ways:

- Writing down your thoughts and organizing them into complete sentences will help you to understand the method of solution better.
- When you look back on homework to study for a test, or later on in another class, you will understand what you were doing on each problem and the mathematics behind it.
- Other people (teacher, classmates, grader,...) will understand what you are doing at each step, and why you are doing it. This way, you won't lose points for skipping steps or solving the problem in an unusual way.
- Communicating your work will be essential in whatever field you choose. Even though the fields are stereotypically weak on writing, engineers and scientists spend a surprising amount of time writing reports and giving oral presentations.

Examples:

How to put an equation into a sentence:

Good: The derivative of the curve $y = 3\sqrt{x}$ is $\frac{dy}{dx} = \frac{3}{2\sqrt{x}}$.
Bad: *The derivative* $\frac{dy}{dx}$ *is* $\frac{3}{2\sqrt{x}}$ *for the curve* y *which is* $3\sqrt{x}$.
Bad: $\frac{dy}{dx}$ *is the derivative of* $3\sqrt{x}$ *which is* $\frac{3}{2\sqrt{x}}$.

Using sentences incorrectly:

Find an equation for the tangent to the curve $y = x + \frac{2}{x}$ at the point $(1, 3)$.
Bad:

$$\frac{dy}{dx} = 1 - \frac{2}{x^2}$$

so the equation for the derivative is

$$\frac{dy}{dx} = -1 = m$$

so the slope at (x, y) *can be found by* $y - 3 = -1(x - 1)$ *so* $y = -x + 4$ *is the point-slope equation.*

Good:

We first check that $(1, 3)$ is a point on the curve by plugging these values in: $3 = 1 + 2/1$. The derivative of the curve $y = x + \frac{2}{x}$ is

$$\frac{dy}{dx} = 1 - \frac{2}{x^2}. \tag{1}$$

A line tangent to the curve at the point $(1, 3)$ will have slope

$$\left.\frac{dy}{dx}\right|_{x=1} = 1 - \frac{2}{(-1)^2} = -1.$$

Using the point-slope formula with $m = -1$, $x_0 = 1$, and $y_0 = 3$ gives the formula for the line $y - 3 = -1(x - 1)$. Solving for y and simplifying gives

$$y = -x + 4.$$

This is the equation of the line tangent to the curve at that point.

Vague and uninformative sentences

Bad: *We use calculus to find that $y = 3x^2 + 1$ has a slope of 3 at $x = 1/2$.*
Good: To find the slope of the curve $y = 3x^2 + 1$ at the point $x = 1/2$ we find $\frac{dy}{dx}$ evaluated at $x = 1/2$.

$$\begin{aligned} \frac{dy}{dx} &= 6x, \\ \left.\frac{dy}{dx}\right|_{x=1/2} &= 6\left(\frac{1}{2}\right) = 3. \end{aligned}$$

Bad:

$$\begin{aligned} \frac{d}{dx}\left((x^2+1)(x^3+3)\right) &= (x^2+1)3x^2 + (x^3+3)2x \\ &= 5x^4 + 3x^2 + 6x \end{aligned}$$

is the derivative.
Good: To take the derivative of a product of 2 functions, we use the product rule, $(fg)' = f'g + fg'$. In our case we have

$$\begin{aligned} \frac{d}{dx}\left((x^2+1)(x^3+3)\right) &= \left(\frac{d}{dx}(x^2+1)\right)(x^3+3) + (x^2+1)\left(\frac{d}{dx}(x^3+3)\right) \\ &= (2x)(x^3+3) + (x^2+1)(3x^2) \\ &= 5x^4 + 3x^2 + 6x. \end{aligned}$$

Referring to previous equations and figures.

You may have noticed that one of the equations above has been labeled equation number one by putting "(1)" towards the right hand margin. If you need to refer back to an equation or figure, label it and then refer to it by its label. Do not draw arrows.
Bad: *Using the equation from before, the slope is -1.* Which equation?
Good: Using (1), the derivative at $x = 1$ is -1.

Mathematical Symbols

You will encounter many mathematical symbols during your math courses. The table below provides you with a list of the more common symbols, how to read them, and notes on their meaning and usage. The following page has a series of examples of these symbols in use.

Symbol	**How to read it**	**Notes on meaning and usage**
$a = b$	a equals b	a and b have exactly the same value.
$a \approx b$ or $a \cong b$	a is approximately equal to b	Do not write $=$ when you mean $\approx$.
$P \Rightarrow Q$	P implies Q	If P is true, then Q is also true.
$P \Leftarrow Q$	P is implied by Q	If Q is true, then P is also true.
$P \Leftrightarrow Q$ or P iff Q	P is equivalent to Q or P if and only if Q	P and Q imply each other.
(a, b)	the point a b	A coordinate in $\mathbb{R}^2$.
(a, b)	the open interval from a to b	The values between a and b, but not including the endpoints.
$[a, b]$	the closed interval from a to b	The values between a and b, including the endpoints.
$(a, b]$	The (half-open) interval from a to b excluding a, and including b.	The values between a and b, excluding a, and including b. Similar for $[a, b)$.
$\mathbb{R}$ or $\mathbf{R}$	the real numbers	It can also be used for the plane as $\mathbb{R}^2$, and in higher dimensions.
$\mathbb{C}$ or $\mathbf{C}$	the complex numbers	$\{a + bi : a, b \in \mathbb{R}\}$.
$\mathbb{Z}$ or $\mathbf{Z}$	the integers	$\ldots,-2,-1,0,1,2,3,\ldots.$
$\mathbb{N}$ or $\mathbf{N}$	the natural numbers	$1, 2, 3, 4, \ldots.$
$a \in B$	a is an element of B	The variable a lies in the set (of values) B.
$a \notin B$	a is not an element of B	
$A \cup B$	A union B	The set of all points that fall in A or B.
$A \cap B$	A intersection B	The set of all points that fall in both A and B.
$A \subset B$	A is a subset of B or A is contained in B	Any element of A is also an element of B.
$\forall x$	for all x	Something is true for all (any) value of x (usually with a side condition like $\forall x > 0$).
$\exists$	there exists	Used in proofs and definitions as a shortcut.
$\exists!$	there exists a unique	Used in proofs and definitions as a shortcut.
$f \circ g$	f composed with g or f of g	Denotes $f(g(\cdot))$.
$n!$	n factorial	$n! = n(n-1)(n-2)\cdots \times 2 \times 1$.
$\lfloor x \rfloor$	the floor of x	The nearest integer $\leq x$.
$\lceil x \rceil$	the ceiling of x	The nearest integer $\geq x$.
$f = \mathcal{O}(g)$ or $f = O(g)$	f is big oh of g	$\lim_{x\to\infty} \sup_{y>x} \lvert f(y)/g(y)\rvert < \infty$. Sometimes the limit is towards 0 or another point.
$f = o(g)$	f is little oh of g	$\lim_{x\to\infty} \sup_{y>x} \lvert f(y)/g(y)\rvert = 0$.
$x \to a^+$	x goes to a from the right	x is approaching a, but x is always greater than a. Similar for $x \to a^-$.

1. **An algebra example:**

$$(y-0)=2(x-1) \quad\Longrightarrow\quad y=2x-2$$

The statement to the left of the $\Longrightarrow$ implies the statement to the right of the $\Longrightarrow$.

2. **Definition: A Formal Definition of Limit**

Let $f(x)$ be defined on an open interval about x_0, except possibly at x_0 itself. We say that $f(x)$ approaches the limit L as x approaches x_0, and we write

$$\lim_{x\to x_0} f(x) = L$$

if for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x

$$0 < |x-x_0| < \delta \Longrightarrow |f(x)-L| < \epsilon.$$

The last statement says $0 < \lvert x-x_0\rvert < \delta$ implies $\lvert f(x)-L\rvert < \epsilon$ is true.

3. **The previous definition can be stated in mathematical symbols,**

The definition of the limit of $f(x)$ as x goes to x_0 ($f(x)\to L$ as $x\to x_0$) is:

if $\forall\epsilon > 0$, $\exists\delta > 0$ s.t. $\forall x$,

$$0 < |x-x_0| < \delta \Longrightarrow |f(x)-L| < \epsilon.$$

Although this statement is correct mathematically, it is difficult to read (unless you are well-versed in math-speak). This example shows that although you can write math is all symbols as a shortcut, often it is clearer to use words.

4. **Theorem: One-sided vs. Two-sided Limits**

A function $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits are equal:

$$\lim_{x\to c} f(x) = L \quad\Leftrightarrow\quad \text{both } \lim_{x\to c^-} f(x) = L \quad\text{and}\quad \lim_{x\to c^+} f(x) = L.$$

The above says that the statement to the left of the $\Leftrightarrow$ is equivalent to the statement on the right hand side. Another way to look at this is that the statement on the left implies the statement on the right, and conversely, the statement on the right implies the statement on the left.

5. **Using $\Leftrightarrow$**

For sets of numbers on the real number line the following two columns are equivalent:

$$\begin{aligned} x\in(a,b) &\quad\Leftrightarrow\quad a<x<b \\ x\in[a,b] &\quad\Leftrightarrow\quad a\le x\le b \\ x\in[a,b) &\quad\Leftrightarrow\quad a\le x<b \\ x\in(a,b] &\quad\Leftrightarrow\quad a<x\le b \end{aligned}$$

Logical Connectives

Mathematics has its own language. As with any language, effective communication depends on logically connecting components. Even the simplest "real" mathematical problems require at least a small amount of reasoning, so it is very important that you develop a feeling for *formal (i.e. mathematical) logic*.

Consider, for example, the two sentences "There are 10 people waiting for the bus" and "The bus is late." What, if anything, is the logical connection between these two sentences? Does one *logically imply* the other? Similarly, the two mathematical statements "$r^2 + r - 2 = 0$" and "$r = 1$ or $r = -2$" need to be connected, otherwise they are merely two random statements that convey no useful information.

Words and symbols that tie statements together logically are called *logical connectives*. They allow you to communicate the reasoning that has led you to your conclusion. Possibly the most important of these is *implication* — the idea that the next statement is a logical consequence of the previous one. This concept can be conveyed by the use of words such as: therefore, hence, and so, thus, since, if ... then ..., this implies, etc. In the middle of mathematical calculations, we can represent these by the implication symbol ($\Rightarrow$). For example

$$x + 7y^2 = 3 \quad \Rightarrow \quad y = \pm\sqrt{\frac{3-x}{7}}\,; \tag{2}$$

$$x \in (0,\infty) \quad \Rightarrow \quad \cos(x) \in [-1,1]. \tag{3}$$

Note that "statement A $\Rightarrow$ statement B" does **not** necessarily mean that the *logical converse* — "statement B $\Rightarrow$ statement A" — is also true. Logical implication is a matter of cause and effect; the logical converse is simply the reverse cause-effect situation (which may not be true). Warning: when mathematicians talk about implication, it means that one thing **must** be true as a consequence of another; not that it can be true, or might be true sometimes. In example (3) above, given that $x \in (0,\infty)$, we cannot escape the fact that $\cos(x) \in [-1,1]$. It may seem that mathematicians are splitting hairs when they quibble over logical details such as these, but consider the following, everyday examples:

- "I lost my pen, therefore I need to get a new one."
- "I am running to class because I am late."

It should be clear that the logical converse of these is not true:

- "I need to get a new pen, therefore I have lost my old one."
- "I am late to class because I am running."

For examples (2) and (3) above:

$$x + 7y^2 = 3 \quad \Leftarrow \quad y = \pm\sqrt{\frac{3-x}{7}}\,;$$

$$x \in (0,\infty) \quad \not\Leftarrow \quad \cos(x) \in [-1,1]. \qquad \boxed{\text{Since } x < 0 \Rightarrow \cos(x) \in [-1,1] \text{ also.}}$$

When the implication works both ways (as in the first example), we say that the two statements are *equivalent* and we may use the equivalence symbol ($\Leftrightarrow$); in words we may say "A is equivalent to B" or "A if and only if B". For example: let X and Y be two sets; if $X = Y$, then $X \subset Y$ and $Y \subset X$; but also, if $X \subset Y$ and $Y \subset X$, then $X = Y$; thus we can say that $X = Y \Leftrightarrow X \subset Y$ and $Y \subset X$.

If two statements are equivalent, we may use any of the implication symbols ($\Rightarrow$, $\Leftarrow$, or $\Leftrightarrow$). Which connective we use depends on what we are trying to show. In (2) above, if we are trying to obtain a formula for y, we would probably just use "$\Rightarrow$", even though the stronger statement ("$\Leftrightarrow$") is also true. If, however, we wanted to show that two statements about x and y were equivalent, we would write $x + 7y^2 = 3 \Leftrightarrow y = \pm\sqrt{\frac{3-x}{7}}$.

We have seen that simply reversing a logical statement can lead to problems. There is a way, however, to invert an implication so that the inverted statement is also true. This is known as the *contrapositive*. Consider the statement "$A \Rightarrow B$"; this means "if A is true, then B is true". The contrapositive is "if B is **false**, then A must be false".

An example of how these concepts fit together:

- If X is a cat, it is an animal. Accept as true
- If X is an animal, it is a cat. FALSE (converse)
- If X is not an animal, it is not a cat. TRUE (contrapositive)
- If X is not a cat, it is not an animal.
  FALSE. This is the contrapositive of the second statement, which is false.

Careful logic is the heart and soul of mathematics; learn to reason with watertight arguments and use logical connectives to explain your reasoning processes.

Example:

What is the greatest amount of water that a right-cylindrical water tank can hold if there is $100m^2$ *of material from which to construct it?*

Good: Let the height of the tank be h and the radius be r. Then the volume of the tank is $V(h,r) = \pi h r^2$ and the surface area is $S(h,r) = 2\pi r^2 + 2\pi r h = 2\pi r(r+h)$. Since we require $V > 0$, we can assume that $r, h > 0$. If we have $100m^2$ of material, then $S = 100$, and so

$$\begin{aligned} S &= 2\pi r(r+h) = 100 \\ &\Rightarrow r + h = \frac{100}{2\pi r} \\ &\Rightarrow h = \frac{50}{\pi r} - r \\ &\Rightarrow V = \pi h r^2 = 50r - \pi r^3. \end{aligned}$$

To find the maximum volume, we look for r such that $\frac{dV}{dr} = 0$. Differentiating with respect to r gives $\frac{dV}{dr} = 50 - 3\pi r^2$. Hence,

$$\begin{aligned} \frac{dV}{dr} = 0 \quad &\Leftrightarrow \quad 50 - 3\pi r^2 = 0 \\ &\Leftrightarrow \quad r^2 = \frac{50}{3\pi} \\ &\Leftrightarrow \quad r = \pm\sqrt{\frac{50}{3\pi}}. \end{aligned}$$

Since $r > 0$, we can use the second derivative test and find $\frac{d^2V}{dr^2} = -6\pi r < 0$. This implies that $r = \sqrt{\frac{50}{3\pi}} \approx 2.3033$ maximizes V. From above, the optimal volume is

$$\begin{aligned} V &= 50r - \pi r^3 \\ &= \left(50 - \pi\left(\frac{50}{3\pi}\right)\right)\sqrt{\frac{50}{3\pi}} \\ &= \left(\frac{100}{3}\right)\sqrt{\frac{50}{3\pi}} \approx 76.7765. \end{aligned}$$

Thus we have found that the greatest amount of water such a tank can hold is (approximately) $76.7765m^3$.

Graphs

Graphs are an important visual aid in mathematics and it is important to learn how to draw them properly. For a graph to be useful it must display the important and interesting information about a function and have all relevant points clearly labeled.

When drawing graphs, it is helpful to ask yourself:

- Have I graphed what is being asked?
- Have I put a title on the graph and labeled the axes?
- Is the scale I have chosen for my graph appropriate?
- Have I illustrated the important features?
- Have I included enough detail?

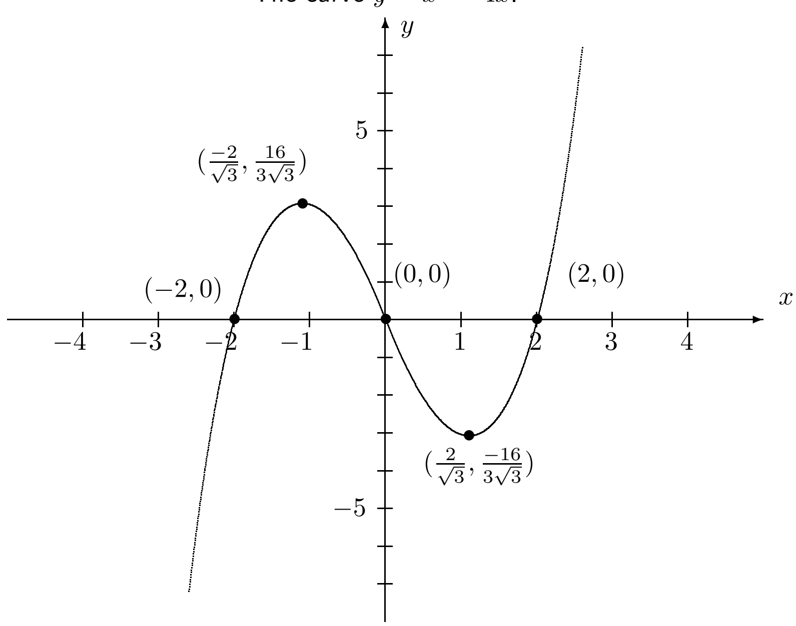
You can often determine how much detail to include in your graph from the question. For example, if you are finding the extreme points of a function, then you will probably be expected to label these points on your graph. If you are only sketching the graph of a function then less detail is required. The main purpose of a sketch is to get some idea of the general shape and behavior of the function. In this case, it is not usually necessary to label extreme points and intercepts. However, you should always remember to label your axes and put a title on your graph, even if it is only a sketch. Use a ruler if necessary to draw straight lines. If you are drawing more than one curve on the same set of axes, take particular care to make sure that each curve is clearly labeled.

Examples:

Graph the curve $y = x^3 - 4x$.

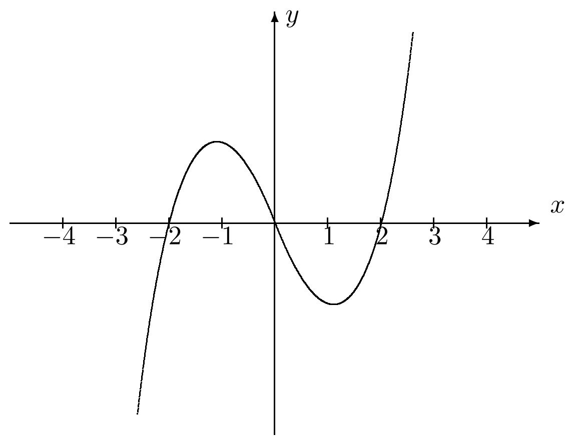
Good:

The curve $y = x^3 - 4x$.



The curve, axes, extreme points, and x- and y-intercepts are clearly labeled. These are the 'points of interest' for this function.

Bad:



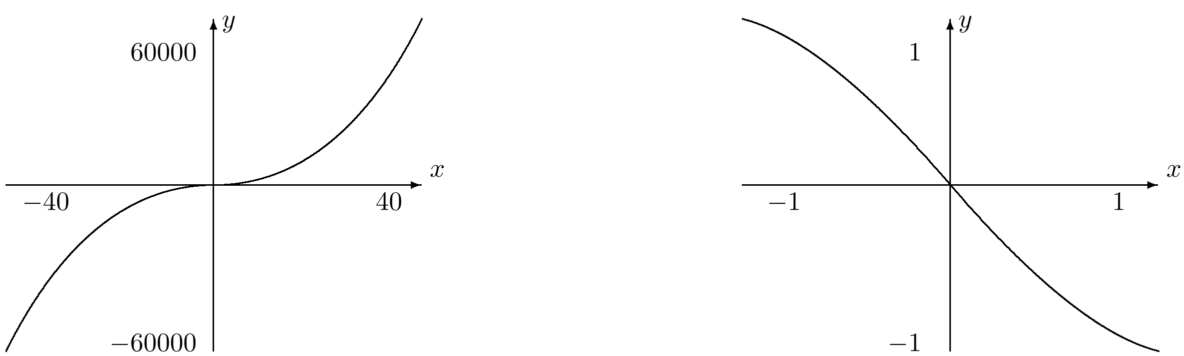
Although this second solution has labeled axes, the graph has no title so it is not clear to someone else what they are viewing. There are also no labels on the extreme points or intercepts and no scale is given on the y-axis. Although this is a bad example of a solution to this question it would be sufficient if only a sketch was required.

Choose a suitable scale so that all of the interesting features of the graph are included. For example, when graphing our function

$$y = x^3 - 4x$$

a suitable scale meant including all of the extreme points and x- and y-intercepts. Be especially careful about the scale when using a graphing calculator or computer. They do not have any idea which points are interesting, so they will often give inappropriate scales. The following graphs give examples of bad choices of scale for our problem.

Bad:



The graph on the left uses too large a scale so the extreme points are hard to see and will be difficult to label. The scale used for the graph on the right is too small as it fails to include two of the x-intercepts.

Also make sure that what you are graphing makes sense. If you are drawing the graph of a function that represents the distance of a space shuttle from the surface of the earth against time, it makes no sense to include negative distances as this is unphysical. Similar reasoning applies to graphs representing populations, areas, and volumes.

Finally, being able to draw and sketch graphs well can sometimes help while trying to answer a question, even if a graph is not required in the solution. For more difficult problems in particular, being able to sketch a function to identify its behavior often provides more insight than looking at the equation alone.

Introductions and Conclusions

Introductions

The purpose of an introduction is to explain the important techniques used to solve your problem. The length may vary from a few sentences to a paragraph. The length usually depends upon the number of steps in your problem needed to get to a solution. A well constructed introduction will make your answer and work easy to follow. On the other hand, a poorly constructed introduction will confuse the reader.

We will illustrate some things to cover in your introduction with the problem: *Use the Sandwich Theorem to find the asymptotes of the curve* $y = \sin(x)/x$.

- What the problem is about. Ask yourself: "What am I doing?" Someone reading your problem should get an idea of what you are about to do. If there are multiple parts explain each part.

  Bad: *I'm going to calculate the asymptotes.*

  Good: I'm going to find the asymptotes of the function $y = \sin(x)/x$.
  This one is supplies more information to the reader.

- The technique(s) you are going to use. This will usually correspond to the topic you are learning. Your assigned problem will generally come from a section you are covering in class, and you can use this as a guide. Sometimes your problem will involve several concepts and you should mention what the predominant technique is that you are going to use. Do not mention the details of your calculations.

  Bad: *Calculating the limit of* $y = \sin(x)/x$ *will tell me where the asymptotes are.*

  Good: To find the asymptotes, I will take the limits as x approaches infinity. The Sandwich Theorem will verify the results.
  The techniques employed are mentioned

- Physical interpretation, if appropriate. Whether this is necessary depends upon the question. If applicable, this step is both helpful for you and the person who is reading your problem. Again, think about what you are doing. Give a physical description of the mathematical steps you are employing to solve your problem. If you were to draw a tangent line at a point, what would the picture look like?

  Bad: *By looking at the graph,* y *goes to the* x *axis.* What are you talking about?

  Good: By looking at the graph of x and y, we see that as x approaches $\pm\infty$, y approaches 0, which is the x axis.
  This description is much clearer.

Putting it all together:
Bad: *I'm going to calculate the asymptotes. Calculating the limit of* $y = \sin(x)/x$ *will tell me where the asymptotes are. By looking at the graph,* y *goes to the* x *axis.*
Good: I'm going to find the asymptotes of the function $y = \sin(x)/x$. To find the asymptotes, I will take the limits as x approaches positive/negative infinity. The Sandwich Theorem will verify the results. By looking at the graph of x and y, we see that as x approaches $\pm\infty$, $y \to 0$, which is the x axis.

The bad introduction doesn't flow very well. Someone reading it would not understand what you did. It is obvious that you calculated the asymptotes but when solving a problem, the techniques employed are as important as your answer. The good introduction briefly explains how you used the Sandwich Theorem along with the result you got. This would allow the reader to have a clear understanding of the mathematical calculations that follow.

Conclusions

Your solution should end with a brief concluding paragraph. The conclusion is a short, concise paragraph that summarizes the main ideas, techniques and results discussed in the problem. The concluding paragraph should focus on the main results and main ideas so that the reader can absorb the important parts of your solution to the problem. The concluding paragraph should not introduce any new material but simply bring back the reader to the main ideas of your solution to the problem. This means that you should not discuss something in the conclusion that was not discussed earlier in your solution.

When you write a conclusion, put yourself in the potential reader's position and imagine what kind of information this reader may look for when he or she reads the concluding paragraph. Also, ask yourself "What do I want my reader to remember?" Here are a few suggestions of what you may want to include in the conclusion paragraph:

- Restate the problem you solved.
- Restate the results and interpretation of the results. If your results are lengthy, such as a big table with numerical values, just describe the results qualitatively. For example, give the range of values for your results.
- Indicate if your results seem reasonable. If not, then explain why.
- If you were unsuccessful in solving your problem, mention possible reasons for this.
- If applicable, mention other problems that are related to the problem you solved.
- Give suggestions for improvements, i.e., what else you could do for the problem.

These are just suggestions, and not all of them may be applicable for your problem. As mentioned above, the conclusion should be short and concise without discussing anything new that was not discussed earlier in the report.

Example Problem: *Find the dimensions of a right circular cylinder of volume* $10^{-6}\ m^3$ *such that a minimum amount of material is used.*

Bad: *In this problem I used calculus to solve a mathematical problem.* This gives no information.

Bad: *I solved for h in* $\pi r^2 h = 10^{-6}$ *and plugged the expression for h into* $2\pi r^2 + 2\pi rh$. *Then I took the derivative of* $2\pi r^2 + \frac{2\times 10^{-6}}{r}$. *Then I set the derivative equal to zero. Then I solved for r.*

The last example fails to focus on the main ideas of the problem. It gives a (bad) summary of some of the steps used to solve the problem. It does not summarize the problem given to us and it does not clearly indicates what your bottom line was or if your were even successful in solving the problem.

Good: The problem discussed in this paper is to find the dimensions of a right circular cylinder of volume $10^{-6}\ m^3$ such that a minimum amount of material is used. The problem was solved by minimizing the area of the cylinder using the first and second derivative test. The optimal dimensions were found to be radius $\cong 5.42\ cm$ and height $\cong 10.84\ cm$. The method used for solving the problem can be applied to other optimization problems, e.g., finding the dimensions of other shapes with a fixed volume such that the used material is minimized. A possible development of the technique used could be to consider optimization of a function of more than two parameters and more than one constraint.

Note that the suggested conclusion requires that we actually did discuss possible applications and developments of the technique earlier in the report, something that may not be required for you in your course.