Math 1300 Spring 08 Review for Exam 1 Solutions.

Concepts

1. (a) False; Domain of $u(x)$ - all real except $x=0$, $x=1$.
   Domain of $v(x)$ - all real except $x=0$.

   (b) False; Let $f(x)=x^2$, $g(x)=\sin x$.
   $(f\circ g)(x) = f(g(x)) = \sin^2 x$
   $\neq$
   $(g\circ f)(x) = g(f(x)) = \sin(x^2)$.

   (c) False;

   (d) True; $\log_b 2 = \log_a 2 / \log_a b < \log_a 2$ since $\log_a b > 1$.

   (e) False; $y=\sqrt{x}$, $\lim_{x\to 0^+}\sqrt{x} = 0$, $\lim_{x\to 0^-}\sqrt{x} = \text{DNE}$

   (f) False; Domain doesn't include $x=0$.

   (g) False; $f(x) = \begin{cases} x & \text{if } x\neq 0 \\ 1 & \text{if } x=0 \end{cases}$; $\lim_{x\to 0} f(x) = 0 \neq f(0)$

   (h) False; Let $f(x) = \begin{cases} x^2+1 & \text{if } x\geq 2 \\ x-3 & \text{if } x<2 \end{cases}$

   (e) False: Let $h(x)=0$, $f(x) = \begin{cases} 1 & \text{if } x\geq 0 \\ 0 & \text{if } x<0 \end{cases}$, $g(x)=1$

   $\lim_{x\to 0} h(x) = 0$ $\quad \lim_{x\to 0} f(x) = \text{DNE}$ $\quad \lim_{x\to 0} g(x) = 1$

Section 1.1

1. (a) $y^3 = 17-2x-x^2 \Rightarrow y = \sqrt[3]{17-2x-x^2}$: YES

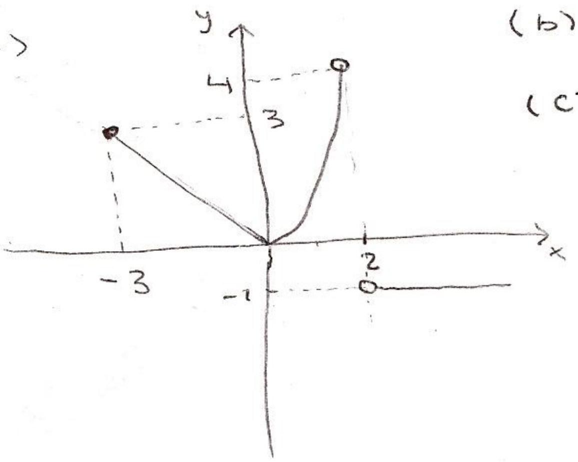
   (b) $(x+1)^2+(y+2)^2 = 6$; For $x=0$, $(y+2)^2=5 \Rightarrow y+2=\pm\sqrt{5}$, $y=-2\pm\sqrt{5}$: NO

   (c) For $x=1$, $y = \pi/2 + 2n\pi$ for any integer n : NO.

2. YES. Domain; 10 missles
   Ranges; targets blown

3. NO, a phrase $\rightarrow$ multiple entries
   $x \longrightarrow y_1, y_2 \cdots$

4. (a)

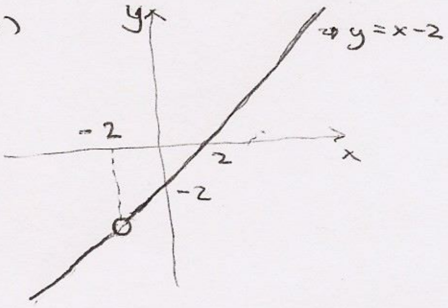


(b) $-3\leq x<2$, $2<x\leq 5$

(c) $0\leq y<4$, $y=-1$

Solutions

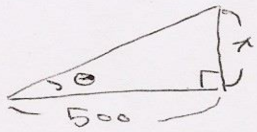
5. (a)



(b) all real except $x=-2$ (OR) $-\infty<x<-2,\ -2<x<\infty$

(c) $-\infty<y<-4,\ -4<y<\infty$

6. (a)



$\tan\theta = \frac{x}{500}$

$x = 500\tan\theta$

(b) $0 \le \theta < \pi/2$

7 (a) NO; $y = \pm\sqrt{3-x^2}$ for $x=0$; $y=\pm\sqrt{3}$

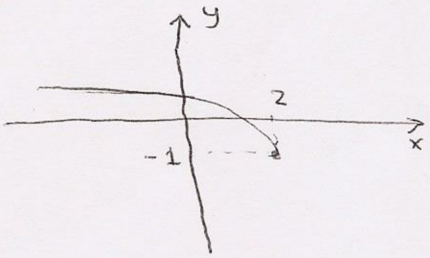
(b) YES; $y = \sqrt[3]{7-x^2}$; Domain - all real x.

(c) YES: Domain - all real x

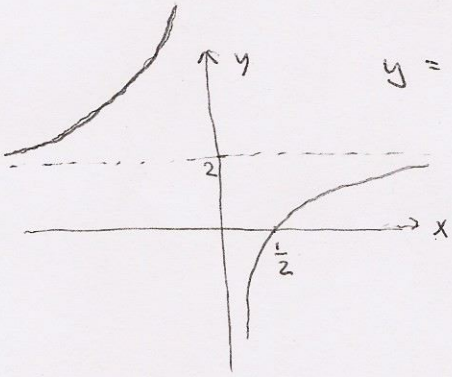
(d) NO: For $x=0$; $y(0)=1$, $g(0)=4-0^3=4$.

SECTION 1.3

1. (a)

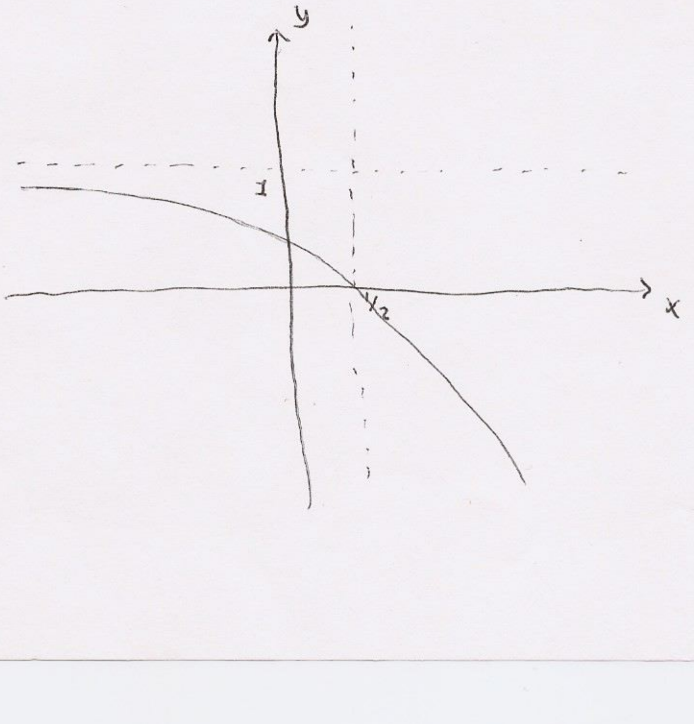


(b)



$y = 2 - \frac{1}{x}$

(c) $y = 1 - e^{2x-1} = 1 - e^{2(x-\frac{1}{2})}$



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Solutions

2. $f(x) = \sqrt{x}$ Domain $x \geq 0$.

$(f \cdot f)(x) = \sqrt{x}\sqrt{x} = x$ with domain $x \geq 0$.

$g(x) = x$ Domain : all real x.

3. $(g \circ f)(x)$ - Domain $(-\infty, \infty)$

$(f \circ g)(x)$ - Domain $(-1, \infty)$

4. $(g \circ f)(x) = g(8-x) = \dfrac{1}{\sqrt{8-x} - 1}$; Domain : $x \leq 8$, $x \neq 7$

$(f \circ g)(x) = f\left(\dfrac{1}{\sqrt{x}-1}\right) = 8 - \left(\dfrac{1}{\sqrt{x}-1}\right)$; Domain : $x \geq 0$, $x \neq 1$

5. $a = -1$, $b = -3$, $c = -4$.

6. $(f \circ g)(x) = f(\ln x) = \sin^{-1}(\ln x)$ Domain $[1/e, e]$

$(g \circ f)(x) = g(\sin^{-1} x) = \ln(\sin^{-1} x)$ Domain $(0, 1]$

SECTION 1.5

(a) $f(x) = \dfrac{x+1}{x-1} = 1 + \dfrac{2}{x-1}$

$\boxed{x = \dfrac{y+1}{y-1} \Rightarrow x(y-1) = y+1 \Rightarrow y(x-1) = x+1 \Rightarrow f^{-1}(x) = \dfrac{x+1}{x-1}}$

Domain of $f(x)$; $x \neq 1$ $\longleftrightarrow$ Range of $f^{-1}(x)$; $y \neq 1$

Range of $f(x)$: $y \neq 1$ $\longleftrightarrow$ Domain of $f^{-1}(x)$; $x \neq 1$.

(b) $f(x) = 3x^3 - 15$ $\boxed{3y^3 = x + 15 \Rightarrow f^{-1}(x) = \sqrt[3]{\dfrac{x+15}{3}}}$

Domain of $f^{-1}(x)$: $(-\infty, \infty)$

Range of $f^{-1}(x)$: $(-\infty, \infty)$

(c) $f(x) = \sqrt{-x}$ $\boxed{x = \sqrt{-y} \Rightarrow x^2 = -y \Rightarrow f^{-1}(x) = -x^2}$

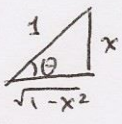
Domain of $f(x)$: $x \leq 0$ $\leftrightarrow$ Range of $f^{-1}(x)$: $y \leq 0$

Range of $f(x)$: $y \geq 0$ $\leftrightarrow$ Domain of $f^{-1}(x)$: $x \geq 0$.

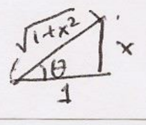
(d) Domain of $f^{-1}(x)$: $(-\infty, \infty)$ $\boxed{x = 12y^3 - 1 \Rightarrow f^{-1}(x) = \sqrt[3]{\dfrac{x+1}{12}}}$

Range of $f^{-1}(x)$: $(-\infty, \infty)$

2. (a) $\tan(\sin^{-1} x) = x / \sqrt{1-x^2}$



(b) $\cos(\tan^{-1} x) = 1 / \sqrt{1+x^2}$



Solutions

SECTION 1.6.

2. (a) $\ln(x^2) = 2\ln|x|$.

(b) $(-27)^{2/3} = ((-3)^3)^{2/3} = 9$

(c) $e^{3\ln\pi} = x \Rightarrow 3\ln\pi \underbrace{\ln e}_{1} = \ln x \Rightarrow x = \pi^3$

(d) $\log_4(\tfrac{1}{2}) + \log_4(8) + \log_4(16) = \underbrace{\log_4(1)}_{\text{zero}} - \log_4 2 + 3\log_4 2 + 2\log_4 4$

$= 2\log_4 2 + 2\log_4 4 = 1 + 2 = 3.$

3. (a) $\ln(x^2) = \ln x \Rightarrow x > 0$

$2\ln|x| = \ln x$

$2\ln x = \ln x$

$\ln x = 0$

$x = 1$

(b) $\ln(x^2) = (\ln x)^2 \Rightarrow x > 0$

$2\ln|x| = (\ln x)^2$

$2\ln x = (\ln x)^2$

$\ln x = 0$ or $\ln x = 2$

$x = 1, e^2$

(c) $\ln\sqrt{x} = \sqrt{\ln x} \Rightarrow x > 0, \ln x \geq 0 \Leftrightarrow x \geq 1$

$\frac{1}{2}\ln x = \sqrt{\ln x}$

$\ln x = 0$ or $\sqrt{\ln x} = 2$

$x = 1, e^4$

(d) $5^x = 4$

$x\ln 5 = \ln 4$

$x = \dfrac{\ln 4}{\ln 5}$

(e) $2^{4x+1} = 3$

$$(4x+1)\ln 2 = \ln 3$$

$$4x+1 = \frac{\ln 3}{\ln 2}$$

$$x = \frac{\frac{\ln 3}{\ln 2} - 1}{4}$$

(f) Let $y = \cos x$

$$y^2 - y + 2 = 0$$

$$y = \frac{1 \pm \sqrt{-7}}{2}$$ No real solution

4. $2^{t/\tau} = e^{\alpha t}$

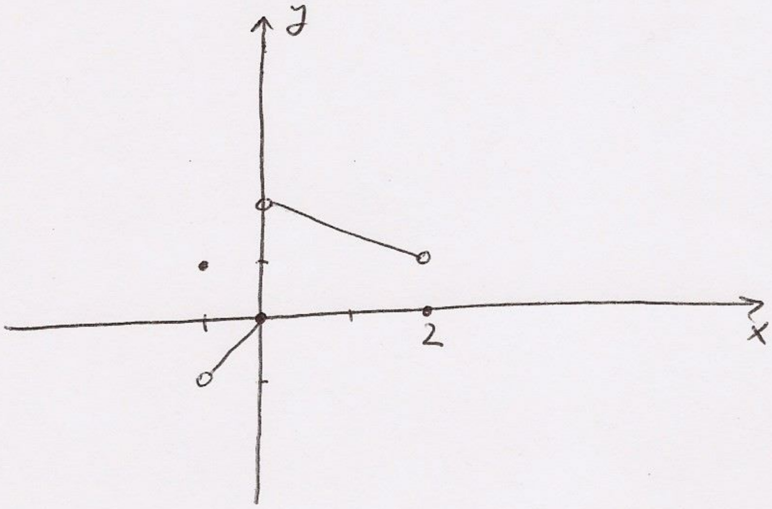
$$\frac{t}{\tau}\ln 2 = \alpha t$$

$$\alpha = \frac{\ln 2}{\tau}$$

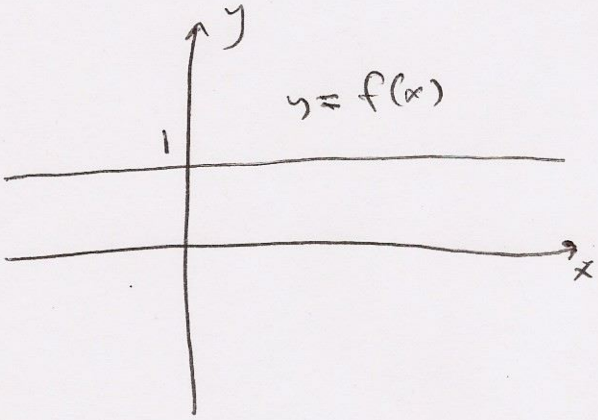
5. $y = \log_b x = \dfrac{\ln x}{\ln b}$

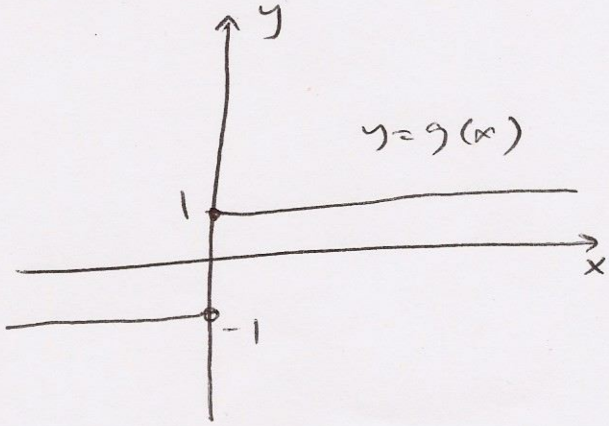
Section 2.1

(1)

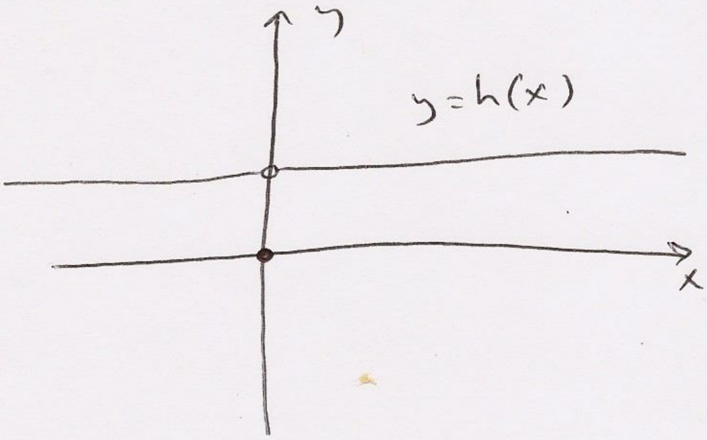


(2) (a)





(c)



Section 2.2

1. (a) $\lim\limits_{x\to 2} x^2+4x-12 = 4+8-12=0$

(b) $\lim\limits_{x\to 2} \dfrac{x^2+4x-12}{x^2+4x+3} = \dfrac{0}{15} = 0$

(c) $\lim\limits_{x\to 2} \dfrac{x^2+4x-12}{x^2-x-2} = \lim\limits_{x\to 2} \dfrac{(x+6)(x-2)}{(x+1)(x-2)} = \lim\limits_{x\to 2} \dfrac{x+6}{x+1} = \dfrac{8}{3}$

(d) $\lim\limits_{x\to 2} \dfrac{x^2+4x-12}{x^2-4x+4} = \lim\limits_{x\to 2} \dfrac{(x+6)(x-2)}{(x-2)^2} = \lim\limits_{x\to 2} \dfrac{x+6}{x-2}$ DNE

(e) $\lim\limits_{x\to -3} \dfrac{x}{x+3}$ DNE

(f) $\lim\limits_{x\to 4} \dfrac{x-4}{\sqrt{x}-2} = \lim\limits_{x\to 4} \dfrac{(\sqrt{x}-2)(\sqrt{x}+2)}{\sqrt{x}-2} = \lim\limits_{x\to 4} \sqrt{x}+2 = 4$

(g) $\lim\limits_{x\to -2} \dfrac{x^2-5x-14}{x+2} = \lim\limits_{x\to -2} \dfrac{(x+2)(x-7)}{x+2} = \lim\limits_{x\to -2} x-7 = -9$

(h) $\lim\limits_{x\to 0} \dfrac{4x-3}{4x^2+3} = \dfrac{-3}{3} = -1$

(i) $\lim\limits_{x\to 3^-} \dfrac{x^2+3x+2}{x^2-2x-3} = \infty$ (DNE)

② (a) $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 2x+1 = 1$

(b) $\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} 2x+1 = 9$

$\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} x^2 = 16$

Since $\lim_{x\to 4^-} f(x) \neq \lim_{x\to 4^+} f(x)$, $\lim_{x\to 4} f(x)$ DNE

③ (a) $\lim_{x\to a}(f(x)+2g(x)) = -3+(2)(6) = 9$

(b) $\lim_{x\to a} \frac{(g(x))^2}{f(x)+5} = \frac{6^2}{-3+5} = \frac{36}{2} = 18$

(c) $\lim_{x\to a} \frac{2f(x)}{h(x)}$ DNE

(d) $\lim_{x\to a} \frac{7f(x)}{2f(x)+g(x)}$ DNE

(e) $\lim_{x\to a} \sqrt[3]{g(x)+2} = \sqrt[3]{6+2} = 2$

Section 2.3

① (a) $\lim_{x\to\infty} 5x^2-2x+1 = \infty$

(b) $\lim_{x\to-\infty} \frac{\sqrt{3x^2-5}}{x-7} = \lim_{x\to-\infty} \frac{\frac{\sqrt{3x^2-5}}{x}}{\frac{x-7}{x}}$

$= \lim_{x\to-\infty} \frac{-\sqrt{\frac{3x^2-5}{x^2}}}{1-\frac{7}{x}} = \lim_{x\to-\infty} \frac{-\sqrt{3-\frac{5}{x^2}}}{1-\frac{7}{x}} = -\sqrt{3}$

(c) $\lim_{x\to\infty} \frac{2}{\pi}\tan^{-1}x = \left(\frac{2}{\pi}\right)\left(\frac{\pi}{2}\right) = 1$

(d) $\lim_{x\to\infty} \frac{\sqrt{5x^2-2}}{x+3} = \lim_{x\to\infty} \frac{\sqrt{\frac{5x^2-2}{x^2}}}{\frac{x+3}{x}}$

$= \lim_{x\to\infty} \frac{\sqrt{5-\frac{2}{x^2}}}{1+\frac{3}{x}} = \sqrt{5}$

(e) $\lim_{x\to\infty} \sqrt{x^2+3} - x = \lim_{x\to\infty} \frac{(\sqrt{x^2+3}-x)(\sqrt{x^2+3}+x)}{\sqrt{x^2+3}+x}$

$= \lim_{x\to\infty} \frac{x^2+3-x^2}{\sqrt{x^2+3}+x} = \lim_{x\to\infty} \frac{3}{\sqrt{x^2+3}+x} = 0$

(f) $\lim_{x\to 1^-} \ln(1-x) = -\infty$

Section 2.5

(1.) $f(x) = \begin{cases} \dfrac{x}{x^2-1}, & x \geq 0 \\ \dfrac{x+3}{|x-3|}, & x < 0 \end{cases} = \begin{cases} \dfrac{x}{x^2-1}, & x \geq 0 \\ -\dfrac{x+3}{x-3}, & x < 0 \end{cases}$

$f(x) = \dfrac{x}{x^2-1}, \; x \geq 0 \Rightarrow f$ is not continuous at $x=1$

$$\left.\begin{aligned} \lim_{x \to 0^+} f(x) &= \lim_{x \to 0^+} \frac{x}{x^2-1} = 0 \\ \lim_{x \to 0^-} f(x) &= \lim_{x \to 0^-} -\frac{x+3}{x-3} = 1 \end{aligned}\right\} f \text{ is not continuous at } x=0$$

$\therefore f$ is discontinuous at $x = 0, 1$

(2) $f(x) = \begin{cases} \sqrt{x^2-16}, & x \geq 5 \\ \dfrac{3k}{x-1}, & x < 5 \end{cases}$

Need f to be continuous at $x=5$:

$$\lim_{x \to 5^-} f(x) = \lim_{x \to 5^-} \frac{3k}{x-1} = \frac{3k}{4}$$

$$\lim_{x \to 5^+} f(x) = \lim_{x \to 5^+} \sqrt{x^2-16} = 3$$

$\therefore \dfrac{3k}{4} = 3$

$k = 4$

3 Let $g(x) = x^5 + \pi x - e$

We have that g is continuous everywhere, since it is a polynomial. Furthermore,

$$g(0) = -e \,, \quad g(1) = 1+\pi - e,$$

$$g(0) = -e < 0 < 1+\pi-e = g(1).$$

The Intermediate Value Theorem implies the existence of $c \in [0,1]$ such that $g(c)=0$, i.e. $c^5 + \pi c - e = 0$

Therefore, f is not continuous on $[0,1]$.

SECTION 2.6

1. If $3x \le f(x) \le x^3 + 2$ for $0 \le x \le 2$, evaluate $\lim_{x \to 1} f(x)$

LET $g(x) = 3x$ and LET $h(x) = x^3 + 2$

NOTE THAT $\lim_{x \to 1} g(x) = \lim_{x \to 1} 3x = 3 \cdot 1 = 3$

AND $\lim_{x \to 1} h(x) = \lim_{x \to 1} (x^3 + 2) = 1^3 + 2 = 3$

SINCE $\underset{\swarrow \, 3}{g(x)} \le f(x) \le \underset{\searrow \, 3}{h(x)}$

AND SINCE $g(x) \to 3$ as $x \to 1$

AND $h(x) \to 3$ as $x \to 1$,

THEN $\boxed{\lim_{x \to 1} f(x) = 3}$ BY THE SQUEEZE THM (p. 157)

SECTION 2.6

2. Prove that $\lim_{x \to 0} (x^4 \cos(2/x)) = 0$

NOTE: $-1 \le \cos\left(\frac{2}{x}\right) \le 1$ for ALL $x \neq 0$

ALSO $-x^4 \le x^4 \le x^4$

SO $\underbrace{-x^4}_{\to 0} \le x^4 \cos\left(\frac{2}{x}\right) \le \underbrace{x^4}_{\to 0}$ for ALL $x \neq 0$

MOREOVER, $x^4 \to 0$ as $x \to 0$ AND $-x^4 \to 0$ as $x \to 0$

SO $\boxed{\lim_{x \to 0} x^4 \cos\left(\frac{2}{x}\right) = 0}$ BY THE SQUEEZE THM

(p. 157)

SECTION 3.1

1. Let $f(x) = x^2 - x$

(a) Find the slope of the tangent line at $x = 2$ using the limit definition of m_{tan}.

(b) Find the equation of the tangent line to the graph of $f(x)$ at $x = 2$.

$$m_{TAN} = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$ IS THE SLOPE OF THE TAN LINE TO THE GRAPH OF f AT THE PT $(x_0, f(x_0))$

FOR $f(x) = x^2 - x$, WE HAVE $f(x_0+h) = (x_0+h)^2 - (x_0+h)$

SO $$m_{TAN} = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{h \to 0} \frac{[(x_0+h)^2 - (x_0+h)] - [x_0^2 - x_0]}{h}$$

$$= \lim_{h \to 0} \frac{[x_0^2 + 2x_0h + h^2 - x_0 - h] - [x_0^2 - x_0]}{h} = \lim_{h \to 0} \frac{x_0^2 + 2x_0h + h^2 - x_0 - h - x_0^2 + x_0}{h}$$

$$= \lim_{h \to 0} \frac{2x_0h + h^2 - h}{h} = \lim_{h \to 0} \frac{h(2x_0 + h - 1)}{h} = \lim_{h \to 0} (2x_0 + h - 1)$$

$$= 2x_0 + 0 - 1 = 2x_0 - 1$$

SO $m_{TAN} = 2x_0 - 1$ IS THE SLOPE OF THE TAN LINE AT $(x_0, f(x_0))$.

AT $x_0 = 2$: $\boxed{m_{TAN} = 2x_0 - 1 = 2 \cdot 2 - 1 = 3}$

AT $x_0 = 2$: $y_0 = f(2) = 2^2 - 2 = 4 - 2 = 2$

SO THE PT OF TANGENCY IS $(x_0, y_0) = (2, 2)$

THE EQUATION OF THIS LINE IS: $y - y_0 = m_{TAN}(x - x_0)$

$$\boxed{y - 2 = 3(x - 2)}$$

SECTION 3.1

2. Let g(x) = |x|. Using the limit definition of m_{tan}, prove that there is not a tangent line to the graph of g at x = 0 (show that the slope of the tangent line does not exist at x = 0).

$$m_{TAN} = \lim_{h \to 0} \frac{g(x_0+h) - g(x_0)}{h}$$ IS THE SLOPE OF THE TAN LINE TO THE GRAPH of g AT THE PT $(x_0, f(x_0))$

FOR $g(x) = |x|$, $m_{TAN} = \lim_{h \to 0} \frac{|x_0+h| - |x_0|}{h}$

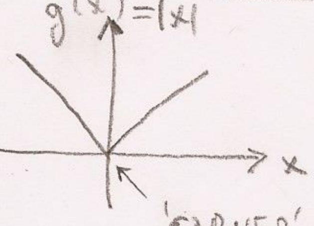
FOR $x_0 = 0$, $m_{TAN} = \lim_{h \to 0} \frac{|0+h| - |0|}{h} = \lim_{h \to 0} \frac{|h|}{h}$

BUT OBSERVE:

$$\lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0^+} \frac{h}{h} = \lim_{h \to 0^+} 1 = 1$$

AND $$\lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0^-} \frac{-h}{h} = \lim_{h \to 0} -1 = -1$$

DEF: $|h| = \begin{cases} h, & h \geq 0 \\ -h, & h < 0 \end{cases}$



'CORNER'

SINCE THE TWO 1-SIDED LIMITS ARE NOT EQUAL, THE 2-SIDED LIMIT DOES EXIST,

i.e. $\lim_{h \to 0} \frac{|h|}{h} = \text{DNE}$. SO THE SLOPE OF THE TAN LINE TO THE GRAPH OF $g(x) = |x|$ AT THE PT $(0, |0|) = (0, 0)$ DOES NOT EXIST, SO THERE IS NO TAN LINE AT $(0,0)$.

INFORMALLY, $g(x) = |x|$ DOES NOT HAVE A TAN LINE AT (0,0) BECAUSE OF THE 'CORNER' AT (0,0)