Math 1300: Calculus I, Spring 2008
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Solution to Exercise 69 in Section 3.6

Let

$$f(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

(a) Show that f is continuous at $x = 0$.

Solution: To show that f is continuous at $x = 0$, we need to show

$$\lim_{x \to 0} f(x) = f(0).$$

First, note that

$$-1 \leq \sin \frac{1}{x} \leq 1.$$

This implies that

$$-|x| \leq x \sin \frac{1}{x} \leq |x|.$$

But we see that

$$\lim_{x \to 0} -|x| = 0 = \lim_{x \to 0} |x|.$$

By the Squeeze Theorem, we must have

$$\lim_{x \to 0} x \sin \frac{1}{x} = 0,$$

which is exactly what $f(0)$ is. Therefore, f is continuous at $x = 0$.

(b) Use Definition 3.2.1 to show that $f'(0)$ does not exist.

Solution: Note that Definition 3.2.1 is the limit definition of the derivative from Section 3.2. We see that

$$\begin{aligned} \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} &= \lim_{h \to 0} \frac{h \sin \frac{1}{h} - 0}{h} \\ &= \lim_{h \to 0} \frac{h \sin \frac{1}{h}}{h} \\ &= \lim_{h \to 0} \sin \frac{1}{h}, \end{aligned}$$

but this limit does not exist. Therefore, $f'(0)$ does not exist.

(c) Find $f'(x)$ for $x \neq 0$.

Solution: Using the Product Rule and the Chain Rule, we get (for $x \neq 0$)

$$\begin{aligned} f'(x) &= 1 \cdot \sin(1/x) + x \cdot \cos(1/x) \cdot (-1/x^2) \\ &= \sin(1/x) - \frac{\cos(1/x)}{x}. \end{aligned}$$

(d) Determine whether $\lim_{x \to 0} f'(x)$ exists.

Solution: The answer is: the limit does not exist. Since $\lim_{x \to 0} \sin(1/x)$ does not exist, the limit of the last line above does not exist, and hence, the limit as $x \to 0$ of $f'(x)$ does not exist. The problem is that the graph of f oscillates an infinite number of times "near" $x = 0$. So, the tangent lines near the origin can't decide whether they want to have positive or negative slope.