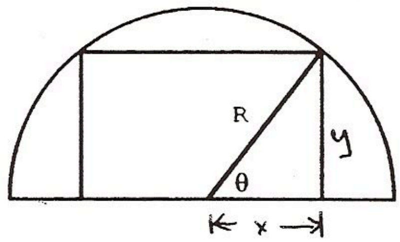
WORKSHEET 10 MATH 1300 March 20, 2008

Goal: To practice solving optimization problems involving trig functions.

1. What is the area of the largest rectangle that can be inscribed in a semicircle of radius R so that one of the sides of the rectangle lies on the diameter of the semicircle.



First solution. Solve this problem by using the length of one side of the rectangle as a variable. (That is, first express the area of the rectangle as a function of the variable x.) Explain how you know that the area you found is maximal.

The area of the square is $A = 2xy$. Since the point (x, y) is on the circle of radius R, $x^2 + y^2 = R^2$, so $y = \sqrt{R^2 - x^2}$. Therefore

$$A = 2x\sqrt{R^2 - x^2}, \quad 0 \le x \le R$$

$$\Rightarrow \frac{dA}{dx} = 2x\left[\frac{-2x}{2\sqrt{R^2 - x^2}}\right] + 2\sqrt{R^2 - x^2} = \frac{-4x^2 + 2R^2}{\sqrt{R^2 - x^2}}$$

This has a critical point in the interval when

$$-4x^2 + 2R^2 = 0 \quad \Rightarrow \quad x = R/\sqrt{2}$$

The area of the square when $x = R/\sqrt{2}$ equals

$$2\underbrace{(R/\sqrt{2})}_{\text{"x"}}\underbrace{(R/\sqrt{2})}_{\text{"y"}} = R^2$$

(which, checking endpoints, is seen to be a max.)

Second solution to problem 1. Solve this problem by using the angle θ, in the drawing on the previous page, as a variable. (That is, first express the area of the rectangle as a function of the variable θ.)

$$A = 2xy \qquad \cos\theta = \frac{x}{R} \Rightarrow x = R\cos\theta$$

$$\sin\theta = \frac{y}{R} \Rightarrow y = R\cos\theta$$

$$= 2R\cos\theta\, R\sin\theta\,, \qquad 0 \le \theta \le \pi/2$$

$$= R^2(2\cos\theta\sin\theta)$$

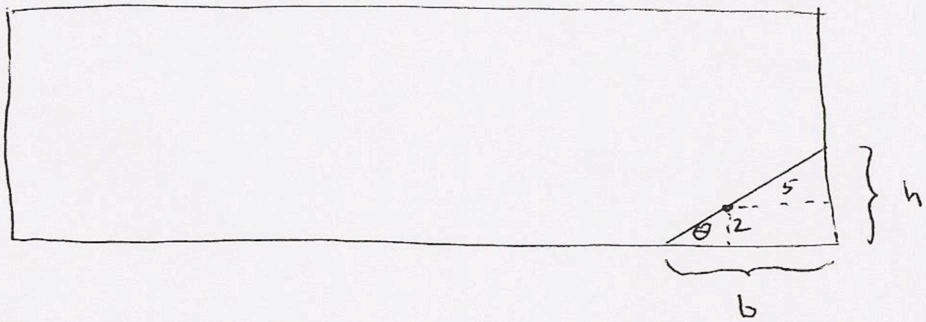
$$= R^2 \sin 2\theta$$

$$\frac{dA}{d\theta} = 2R\cos 2\theta$$ which has a critical point in the interval when $\cos 2\theta = 0$, i.e., $2\theta = \pi/2 \Rightarrow \theta = \pi/4$.

The area of the square is

$$A = R^2 \sin(2\pi/4) = R^2 \sin\pi/2 = R^2.$$

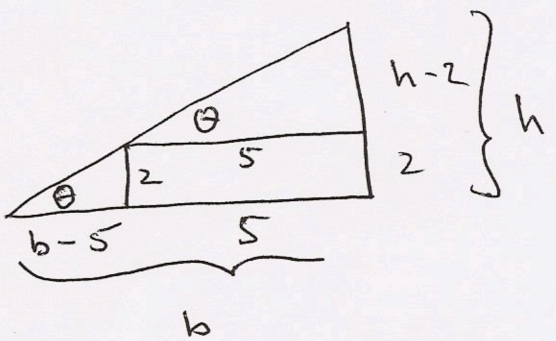
2. Sue finds a tiny cigarette burn in her new rectangular scarf, which measures 5 feet by 2 feet . She decides the best solution is to cut off one corner of the scarf through the burn. The burn is located 2 inches from one edge of the scarf and 5 inches from the end of the scarf.



(a) Show that the area of the triangle may be written as:

$$\frac{1}{2}\Big(2\cot(\theta)+5\Big)\Big(5\tan(\theta)+2\Big).$$

Larger picture



$\text{Area} = \frac{1}{2} b \cdot h$ so we need to relate b & θ & h & θ

$\underline{b \,\&\, \theta}$: $\tan\theta = \frac{2}{b-5} \Rightarrow b-5 = \frac{2}{\tan\theta}$

$\Rightarrow b = 2\cot\theta + 5$

$\underline{h \,\&\, \theta}$ $\tan\theta = \frac{h-2}{5} \Rightarrow h = 5\tan\theta + 2$

These give the above formula

(b) What is the smallest area of the triangle she has to cut off to remove the burn? (Assume the burn is a small dot and the cut is to pass through the burn.)

$$A = \frac{1}{2}\left(2\cot(\theta)+5\right)\left(5\tan(\theta)+2\right)$$

$$= 10 + 2\cot(\theta) + \frac{25}{2}\tan(\theta)$$

$\frac{dA}{d\theta} = -2\csc^2(\theta) + \frac{25}{2}\sec^2(\theta)$ So $\frac{dA}{d\theta} = 0$ when

$\frac{\csc^2\theta}{\sec^2\theta} = \frac{25}{4}$ i.e., when $\tan^2\theta = 25/4$

Therefore $\tan\theta = 5/2$ and the total area is

$A = \frac{1}{2}\left(2\left(\frac{2}{5}\right)+5\right)\left(5\left(\frac{5}{2}\right)+2\right) = \frac{29 \cdot 29}{20}$ In class: Why is this a min?