Goal: To investigate the role of the sign of the first derivative of a function in helping understand its graph

Definition. Let f be a function defined on an interval, and let x_1 and x_2 denote points in that interval.

(a) f is *increasing* on the interval if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$.
(b) f is *decreasing* on the interval if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$.
(c) f is *constant* on the interval if $f(x_1) = f(x_2)$ whenever $x_1 < x_2$.

1. Suppose f is a function that is increasing on an interval and x_1 and x_2 are points in the interval. Determine whether the slope of the secant line through the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ is positive, negative, or zero.

WLG ASSUME $x_1 < x_2$, THEN $f(x_1) < f(x_2)$

SLOPE OF SECANT LINE : $\dfrac{f(x_1) - f(x_2)}{x_1 - x_2} = \dfrac{NEG}{NEG}$

$= $ POSITIVE

2. In the description of f in problem 1, now assume the function if *decreasing* (resp. *constant*) on an interval, and determine whether the slope of the secant line is positive, negative, or zero.

DECREASING:

$x_1 < x_2 \rightarrow f(x_1) > f(x_2)$

SLOPE : $\dfrac{f(x_1) - f(x_2)}{x_1 - x_2} = \dfrac{POS}{NEG} = $ NEGATIVE

CONSTANT:

$x_1 < x_2 \rightarrow f(x_1) = f(x_2)$

SLOPE : $\dfrac{f(x_1) - f(x_2)}{x_1 - x_2} = \dfrac{ZERO}{NEG} = $ ZERO

The results discovered for secant lines in problems 1 and 2 carry over to tangent lines. Specifically, we have the following theorem:

Theorem. Let f be a function that is continuous on a closed interval $[a, b]$, and differentiable on the open interval (a, b).

(a) If $f'(x) > 0$ for every value of x in $[a, b]$ then f is *increasing* on $[a, b]$.
(b) If $f'(x) < 0$ for every value of x in $[a, b]$ then f is *decreasing* on $[a, b]$.
(c) If $f'(x) = 0$ for every value of x in $[a, b]$ then f is *constant* on $[a, b]$.

3 Use the above theorem to sketch the graph of a function whose derivative satisfies the properties:
  f' is positive on each of the intervals $(-\infty, -1)$ and $(3, \infty)$
  f' is zero at $x = -1, x = 1,$ and $x = 3$
  f' is negative on the intervals $(-1, 1)$ and $(1, 3)$
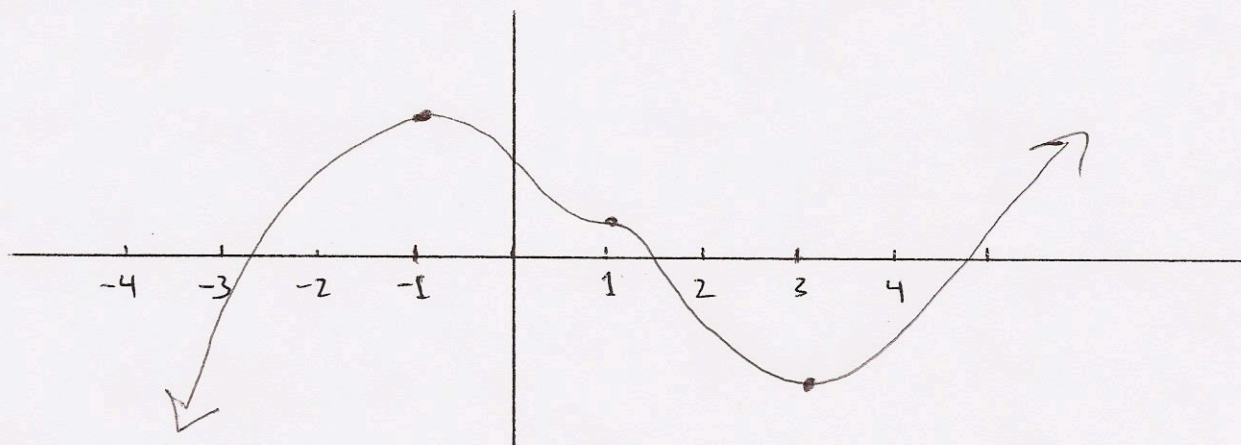
$$\left(\text{OTHER POSSIBLE SOLUTIONS}\ldots\right)$$

4 Let $f(x) = |x - 1| + |x + 2|$. Find where f is increasing, where it is decreasing, and where it is constant. Use this information to help sketch the graph of the function

$$f'(x) = \left(\begin{cases} 1, & x > 1 \\ -1, & x < 1 \end{cases}\right) + \left(\begin{cases} 1, & x > -2 \\ -1, & x < -2 \end{cases}\right)$$

$$= \begin{cases} 2, & x > 1 \\ 0, & -2 < x < 1 \\ -2, & x < -2 \end{cases}$$

INCREASING: $(1, \infty)$; CONSTANT: $(-2, 1)$;

DECREASING: $(-\infty, -2)$