WORKSHEET 9 **MATH 1300** **March 13, 2008**

Goal: To better understand the non-vertical asymptotes of rational function; and, in particular to explore the role of long division in understanding those asymptotes when they are not horizontal.

In this worksheet we examine rational functions, that is functions of the form $f(x) = \dfrac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials. Recall that the degree of a polynomial is the highest power of x appearing in the polynomial, e.g, the degree of the polynomial $P(x) = 2x^4 - 3x + 2$ is 4 while the degree of $Q(x) = x - 3$ is 1.

In this worksheet we will discover that the non-vertical asymptotes of a rational function will depend, in part, on the relationship between the degree of its numerator and the degree of its denominator.

IMPORTANT ASSUMPTION: Henceforth we will only consider rational functions of the form $f(x) = \dfrac{P(x)}{Q(x)}$, where the fraction $\dfrac{P(x)}{Q(x)}$ cannot be simplified, in other words where the polynomials $P(x)$ and $Q(x)$ do not have a common factor.

Preliminary question 1. If you are presented with a rational function $f(x) = \dfrac{P(x)}{Q(x)}$ which can be simplified, why do you think you should simplify it before you try to understand it's graph?

By cancelling as many common factors btwn $P(x)$ and $Q(x)$, we can determine where $f(x)$ has holes and where $f(x)$ has vertical asymptotes.

Preliminary question 2. What, in your own words, is an asymptote? (After you complete each of the problems below compare what you have discovered with your answer to this question.)

$y = a(x)$ is an asymptote of $y = f(x)$ if

$$\lim_{x \to \infty} f(x) = a(x)$$

or

$$\lim_{x \to -\infty} f(x) = a(x).$$

$x = a$ is a vertical asymptote of $y = f(x)$ if

$$\lim_{x \to a^+} f(x) = \infty \text{ or } -\infty$$

or

$$\lim_{x \to a^-} f(x) = \infty \text{ or } -\infty.$$

Case 1. $f(x) = \dfrac{P(x)}{Q(x)}$ where the degree of $Q(x)$ is greater than the degree of $P(x)$.

An example. Let $f(x) = \dfrac{-x^2 - x + 7}{2x^4 - 3x + 2}$

(a) Evaluate each of the limits: $\lim_{x\to\infty} \dfrac{-x^2 - x + 7}{2x^4 - 3x + 2}$ and $\lim_{x\to-\infty} \dfrac{-x^2 - x + 7}{2x^4 - 3x + 2}$.

$$\lim_{x\to\infty} \frac{-x^2-x+7}{2x^4-3x+2} = \lim_{x\to\infty} \frac{-x^2}{2x^4} = \lim_{x\to\infty} \frac{-1}{2x^2} = \boxed{0}$$

$$\lim_{x\to\infty} \frac{-x^2-x+7}{2x^4-3x+2} = \lim_{x\to-\infty} \frac{-x^2}{2x^4} = \lim_{x\to-\infty} \frac{-1}{2x^2} = \boxed{0}$$

(b) What do each of the limits in part (a) tell you about the non-vertical asymptotes of the function

$$f(x) = \frac{-x^2 - x + 7}{2x^4 - 3x + 2}?$$

There is a horizontal asymptote at $y = 0$.

Conclusion. Do you think you will obtain *exactly the same* horizontal asymptote for any rational function $f(x) = \dfrac{P(x)}{Q(x)}$ where the degree of $Q(x)$ is greater than the degree of $P(x)$. (Explain your answer.)

Yes. If $\deg(P(x)) = n$ w/ leading coeff a and $\deg(Q(x)) = m$ w/ leading coeff b, and $m > n$, then

$$\lim_{x\to\infty} \frac{P(x)}{Q(x)} = \lim_{x\to\infty} \frac{ax^n}{bx^m} = \lim_{x\to\infty} \frac{a}{bx^{m-n}} = 0.$$

Similar for $x \to -\infty$.

Case 2. $f(x) = \dfrac{P(x)}{Q(x)}$ where the degree of $Q(x)$ equals the degree of $P(x)$.

An example. Let $f(x) = \dfrac{-x^3 - x + 7}{2x^3 - 3x + 2}$

(a) Evaluate each of the limits: $\displaystyle\lim_{x\to\infty} \frac{-x^3 - x + 7}{2x^3 - 3x + 2}$ and $\displaystyle\lim_{x\to-\infty} \frac{-x^3 - x + 7}{2x^3 - 3x + 2}$.

$$\lim_{x\to\infty} \frac{-x^3 - x + 7}{2x^3 - 3x + 2} = \lim_{x\to\infty} \frac{-x^3}{2x^3} = \boxed{-\frac{1}{2}}$$

$$\lim_{x\to-\infty} \frac{-x^3 - x + 7}{2x^3 - 3x + 2} = \lim_{x\to-\infty} \frac{-x^3}{2x^3} = \lim_{x\to-\infty} \frac{-1}{2} = \boxed{\frac{-1}{2}}$$

(b) What do each of the limits in part (a) tell you about the non-vertical asymptotes of the function

$$f(x) = \frac{-x^3 - x + 7}{2x^3 - 3x + 2}?$$

There is a horizontal asymptote @ $y = \frac{-1}{2}$.

Conclusion. Do you think a rational function $f(x) = \dfrac{P(x)}{Q(x)}$, where the degree of $Q(x)$ equals the degree of $P(x)$, will always have a horizontal asymptote? (Explain your answer.)

Yes. Consider $P(x)$ & $Q(x)$ as on previous pg, ~~except~~ except, assume $m = n$. then

$$\lim_{x\to\infty} \frac{P(x)}{Q(x)} = \lim_{x\to\infty} \frac{ax^n}{bx^m} = \lim_{x\to\infty} \frac{a}{b} = \frac{a}{b}.$$

Similar for $x \to -\infty$. So, horiz asymptote at $y = \frac{a}{b}$.

Case 3. $f(x) = \dfrac{P(x)}{Q(x)}$ where the degree of $Q(x)$ is 1 less than the degree of $P(x)$.

An example. Let $f(x) = \dfrac{x^3 - x}{2x^2 + 2}$

(a) Evaluate each of the limits: $\lim\limits_{x \to \infty} \dfrac{x^3 - x}{2x^2 + 2}$ and $\lim\limits_{x \to -\infty} \dfrac{x^3 - x}{2x^2 + 2}$.

$$\lim_{x \to \infty} \frac{x^3 - x}{2x^2+2} = \lim_{x \to \infty} \frac{x^3}{2x^2} = \lim_{x \to \infty} \frac{x}{2} = \infty$$

$$\lim_{x \to -\infty} \frac{x^3 - x}{2x^2+2} = \lim_{x \to -\infty} \frac{x^3}{2x^2} = \lim_{x \to -\infty} \frac{x}{2} = -\infty$$

(b) What, if anything, do each of the limits in part (a) tell you about possible non-vertical asymptotes of the function

$$f(x) = \frac{x^3 - x}{2x^2 + 2}?$$

There are no ~~nonvert~~ horizontal asymptotes.

However, the line $y = \frac{x}{2}$ is a (slant) asymptote (see below).

How to discover the asymptotes in the above problem. If we divide the numerator by the denominator of $f(x) = \dfrac{x^3 - x}{2x^2 + 2}$ we discover that

$$f(x) = \frac{x^3 - x}{2x^2 + 2} = \frac{1}{2}x + \frac{-2x}{2x^2 + 2}.$$

(c) Calculate each of the limits: $\lim\limits_{x \to \infty} \dfrac{-2x}{2x^2 + 2}$ and $\lim\limits_{x \to -\infty} \dfrac{-2x}{2x^3 + 2}$.

$$\lim_{x \to \infty} \frac{-2x}{2x^2+2} = 0 = \lim_{x \to -\infty} \frac{-2x}{2x^3+2}$$ (by case 1).

(d) Can you explain how your results in (c) allow you to conclude that for x with $|x|$ *very large*

$$f(x) = \frac{x^3 - x}{2x^2 + 2} \approx \frac{1}{2}x.$$

Since $\frac{x^3 - x}{2x^2+2} = \frac{1}{2}x + \frac{-2x}{2x^2+2}$ and

$\frac{-2x}{2x^2+2} \to 0$ as $x \to \infty$ or $x \to -\infty$, we see

that

$$\frac{x^3-x}{2x^2+2} \to \frac{1}{2}x.$$

(e) Evaluate the limit

$$\lim_{x\to\infty} \frac{\frac{x^3-x}{2x^2+2}}{\frac{1}{2}x},$$

and conclude that the approximation becomes better and better as $|x|$ becomes larger and larger. (In other words, the graph of $f(x) = \dfrac{x^3 - x}{2x^2+2}$ becomes closer and closer to the graph of $y = \dfrac{1}{2}x$ as $|x|$ becomes larger and larger.)

$$\lim_{x\to\infty} \frac{\frac{x^3-x}{2x^2+2}}{\frac{1}{2}x} = \lim_{x\to\infty} \frac{\cancel{x^3}^{\,2}-\cancel{x}^{\,1}}{2x^2+2}\cdot\frac{2}{\cancel{x}}$$

$$= \lim_{x\to\infty} \frac{2(x^2-1)}{2x^2+2} = \lim_{x\to\infty} \frac{2x^2}{2x^2} = 1.$$

This implies that in the "tug of war" btwn $\frac{x^3-x}{2x^2+2}$ and $\frac{1}{2}x$, they "tie." In other words, Since the ratio approaches 1, the two functions get closer and closer as $|x|$ gets larger and larger.

Conclusion. Do you think a rational function $f(x) = \dfrac{P(x)}{Q(x)}$, where the degree of $Q(x)$ is 1 less than the degree of $P(x)$, will always have a line as an asymptote? (Explain your answer.)

Yes, in fact, if we take $P(x)$ and $Q(x)$ as before, but this time take $n-m=1$, then

$$\frac{P(x)}{Q(x)} \longrightarrow \frac{ax^n}{bx^m} = \frac{ax^{n-m}}{b} = \frac{ax}{b}$$

as $|x| \to \infty$.

Final question. In tonight's homework you will discover that a rational function can have a *curved* asymptote. How will this discovery force you to reconsider your original definition of an asymptote above?

Asymptotes can be curves!