Section 2.4: The Precise Definition of a Limit (part 1)

Goal

In this section, we will introduce the precise definition of a limit and develop some intuition of this technical definition by looking at examples and a web applet.

Background

First, let's recall the intuitive definition that we gave for a limit back in Section 2.2. We write

$$\lim_{x \to a} f(x) = L$$

and say

"the limit of __________, as x approaches __________, is __________"

if we can make the values of $f(x)$ __________ __________ to L by taking x to be __________ __________ to a (on either side) but not equal to a.

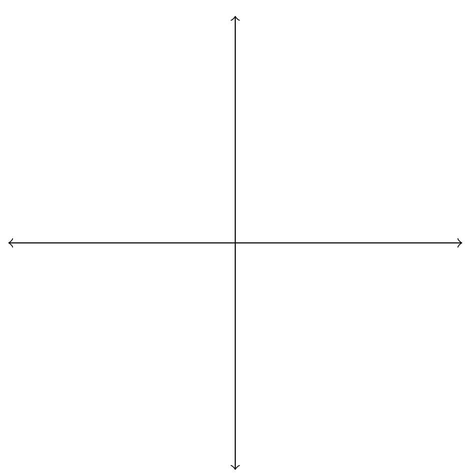
Oftentimes, this intuitive definition is inadequate. Phrases like "x is close to 3" and "$f(x)$ gets closer and closer to 17" are vague. We need to nail down what we mean by "arbitrarily close" and "sufficiently close."

A motivating example

Let's consider the function

$$f(x) = \begin{cases} 3x + 1, & x \neq 2 \\ 5, & x = 2. \end{cases}$$

The graph of this function looks like



Intuitively, we see that when x is close to 2 but $x \neq 2$, $f(x)$ is close to __________, and so

$$\lim_{x \to 2} f(x) = \underline{\qquad\qquad}.$$

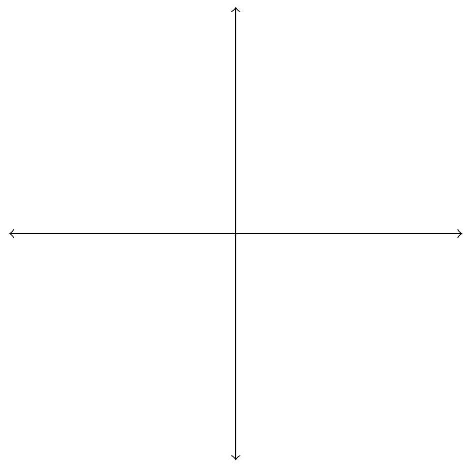
Let's explore what "close" means here. How close to 2 does x have to be so that $f(x)$ differs from ____________ by less than 0.1?

Recall that the distance from x to 2 is ____________ and the distance from $f(x)$ to 7 is ____________. Our goal is to find a distance, which we will call δ, such that

$$\underline{\hspace{3cm}} < 0.1 \text{ if } 0 < \underline{\hspace{3cm}} < \delta.$$

Why did we require $0 < |x - 2|$? Because we are supposed to have ____________.

Here's the corresponding picture



Let's fiddle around and see what we can find out:

What we just learned was that we should take $\delta =$ ____________. That is, if x is within a distance of ____________ from 2, then $f(x)$ will be within a distance of 0.1 from 7.

Now, if change 0.1 to a smaller number like 0.001, then by using a similar argument we could find the corresponding δ (which will be ____________). For 7 to be the precise limit of $f(x)$ as x approaches 2, we must not only be able to bring the difference between $f(x)$ and 7 below 0.1 and 0.001; we must be able to bring it below *any* positive number.

Epsilonics

Definition 1. Let $f(x)$ be a function defined on some open interval that contains the number a, except possibly a itself. Then we say that the *limit of $f(x)$ as x approaches a is L*, and we write

$$\lim_{x \to a} f(x) = L$$

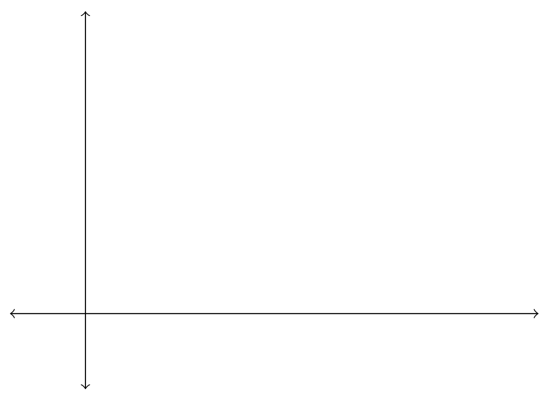
if for every number $\epsilon > 0$, there is a number $\delta > 0$ such that

$$|f(x) - L| < \underline{\hspace{3cm}}$$

whenever

$$0 < |x - a| < \underline{\hspace{3cm}}.$$

Here's the picture:



Let's take a look at some examples using a web applet.

An analogy

We'll think of the $\epsilon - \delta$ definition of the limit as a card game. The game is between a dealer, let's say Plato, and you. There are many hands in the game. There are two ways to win this game, but for now, we'll just talk about one way to win.

In the first way to win you must win every hand. When the game starts, Plato hands you a function and an x-value a. You then pick L. Each hand begins by the dealer handing you an $\epsilon > 0$. This determines an ϵ-tube of y-values, where the tube is perpendicular to the y-axis. (These are the eligible targets in our missile analogy.)

If you can find a $\delta > 0$ such that the part of the function sitting inside intersection of Plato's ϵ-tube and your δ-tube goes out the sides of the corresponding rectangle (and not the top and bottom; the corners are fine), then you win the hand.

You win the whole game if you can win every single hand. That is, you must win each hand for every possible ϵ.

Examples

Let's do some examples where we try to win just a single hand.

Example 2.

(a) Let $f(x) = 4x + 1$ and $\epsilon = 0.1$. Find $\delta > 0$ such that $|f(x) - 5| < \epsilon$ whenever $|x - 1| < \delta$.

(b) Do Exercise 3 on page 95.