Section 2.4: The Precise Definition of a Limit (part 2)

Goal

We will continue to explore the $\epsilon - \delta$ definition of the limit and do a few more examples.

Recall

First, let's remind ourselves what the precise definition of the limit is.

Definition 1. Let $f(x)$ be a function defined on some open interval that contains the number a, except possibly a itself. Then we say that the *limit of* $f(x)$ *as* x *approaches* a *is* L, and we write

$$\lim_{x \to a} f(x) = L$$

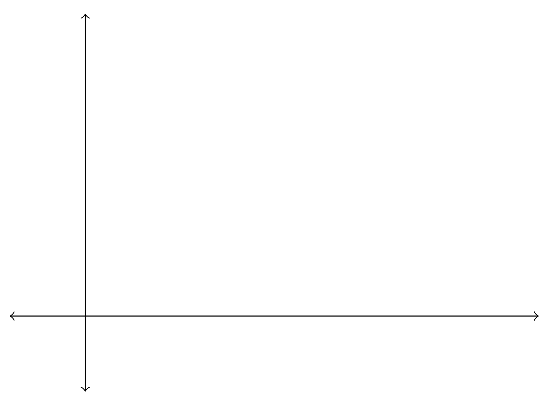
if for every number $\epsilon > 0$, there is a number $\delta > 0$ such that

$$|f(x) - L| < \underline{\hspace{3cm}}$$

whenever

$$|x - a| < \underline{\hspace{3cm}}.$$

Once again, here's the picture:



Important Note 2.

(1) Remember that δ depends on ϵ. When ϵ gets smaller, δ needs to get smaller.

(2) Once you've found a δ that works for a given ϵ, you can always choose a smaller δ.

(3) The goal is to form a 2ϵ by 2δ window around the point (a, L) such that the graph goes out the sides of the box (not the top or bottom; the corners are fine).

So far, we have only looked at examples where ϵ is a particular small positive value. But to prove that the limit of $f(x)$ as x approaches a is L, we need to show that for **every** ϵ, we can find a δ that "works."

More examples

Example 3. Using the $\epsilon - \delta$ definition of the limit, prove that $\lim_{x \to 3} \frac{x^2 - 9}{x - 3} = 6$.

Example 4. Using the $\epsilon - \delta$ definition of the limit, prove that $\lim_{x \to 1} x^2 - x + 2 = 2$.

(Hint for Exercise 39: Suppose the limit does exist and has limit L. Let $\epsilon = 1/2$. If the limit exists, there should be a $\delta > 0$ such that $|f(x) - L| < 1/2$ whenever $|x - 0| < \delta$. Show that for any $\delta > 0$, there is always at least one x-value satisfying $|x - 0| < \delta$, but $|f(x) - L| < 1/2$; this is a contradiction.)