# Partial Solutions to Exam 2

1. (a) $\displaystyle\int \frac{\sin^3 x}{\sqrt{\cos x}}\,dx$

$= \displaystyle\int \sin^2 x \sin x (\cos x)^{-1/2}\,dx$

$= \displaystyle\int (1-\cos^2 x)(\cos x)^{-1/2} \sin x\,dx$

$u = \cancel{\sin x}\ \cos x$
$du = -\sin x\,dx$
$dx = \dfrac{du}{-\sin x}$

$= -\displaystyle\int (1-u^2)\, u^{-1/2}\,du$

$= -\displaystyle\int u^{-1/2} - u^{3/2}\,du$

$= -\left(2u^{1/2} - \dfrac{2u^{5/2}}{5}\right) + C$

$= \boxed{-\left(2\sqrt{\cos x} - \dfrac{2}{5}(\cos x)^{5/2}\right) + C}$

(b) $\displaystyle\int_0^{\sqrt{8}} \frac{x}{\sqrt{9-x^2}}\,dx$

$u = 9 - x^2$
$du = -2x\,dx$
$dx = \dfrac{du}{-2x}$

$= \displaystyle\int_{x=0}^{x=\sqrt{8}} \cancel{x}\, u^{-1/2} \frac{du}{-2\cancel{x}}$

$= \dfrac{-1}{2}\displaystyle\int_{x=0}^{x=\sqrt{8}} u^{-1/2}\,du$

$$= \frac{-1}{\cancel{2}} \cdot \cancel{2}\, u^{1/2} \Big|_{x=0}^{x=\sqrt{8}}$$

$$= -\sqrt{9-x^2} \Big|_{0}^{\sqrt{8}}$$

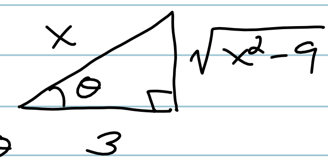
$$= -1 + 3$$

$$= \boxed{2}$$

(c) $\displaystyle\int \frac{\sqrt{x^2-9}}{x}\, dx$ $\qquad x = 3\sec\theta$

$\qquad\qquad\qquad\qquad dx = 3\sec\theta\tan\theta$

$$= \int \frac{\sqrt{9\sec^2\theta - 9}}{\cancel{3}\cancel{\sec\theta}}\, \cancel{3}\,\cancel{\sec\theta} \cdot \tan\theta\, d\theta$$




$$= 3\int \sqrt{\sec^2\theta - 1} \cdot \tan\theta\, d\theta$$

$$= 3\int \tan^2\theta\, d\theta$$

$$= 3\int \sec^2\theta - 1\, d\theta$$

$$= 3\left[\tan\theta - \theta\right] + C$$

$$= \boxed{3\left[\frac{\sqrt{x^2-9}}{3} - \operatorname{arcsec}\left(\frac{x}{3}\right)\right] + C}$$

(d) $\int \frac{x^3}{x^2-9}\,dx$

$$x^2-9 \overline{\big)\, x^3}$$ quotient $x$
$$-(x^3-9x)$$
$$9x$$

$= \int x + \frac{9x}{x^2-9}\,dx$

$u = x^2-9$
$du = 2x\,dx$
$dx = \frac{du}{2x}$

$= \frac{x^2}{2} + 9\int \frac{\cancel{x}}{u}\cdot\frac{du}{2\cancel{x}}$

$= \frac{x^2}{2} + \frac{9}{2}\ln|u| + C$

$= \boxed{\frac{x^2}{2} + \frac{9}{2}\ln|x^2-9| + C}$

(e) $\int \frac{2x^2+3}{x^3+x}\,dx$

$= \int \frac{2x^2+3}{x(x^2+1)}\,dx$

$A(x^2+1) + x(Bx+C) = 2x^2+3$
$Ax^2 + A + Bx^2 + Cx = 2x^2+3$
$(A+B)x^2 + Cx + A = 2x^2+3$

$C = 0$
$A = 3$
$B = -1$

$= \int \frac{A}{x} + \frac{Bx+C}{x^2+1}\,dx$

$= \int \frac{3}{x} + \frac{-x}{x^2+1}\,dx$

$u = x^2+1$
$du = 2x\,dx$
$dx = \frac{du}{2x}$

$= 3\ln|x| - \int \frac{-\cancel{x}}{u}\cdot\frac{du}{2\cancel{x}}$

$= \boxed{3\ln|x| - \frac{1}{2}\ln|x^2+1| + C}$

2. (a) $\displaystyle\int_0^\infty \frac{1}{9x^2+25}\,dx$ (the $\infty$ makes it improper)

$$= \lim_{t\to\infty} \int_0^t \frac{1}{9x^2+25}\,dx$$

$a = 5$
$u = 3x$
$du = 3\,dx$
$dx = \frac{du}{3}$

$$= \lim_{t\to\infty} \frac{1}{3}\cdot\frac{1}{5}\arctan\left(\frac{3x}{5}\right)\Big|_0^t$$

$$= \lim_{t\to\infty} \frac{1}{15}\left[\arctan\left(\frac{3t}{5}\right) - \arctan(0)\right]$$

$$= \frac{1}{15}\left[\frac{\pi}{2} - 0\right]$$

$$= \boxed{\frac{\pi}{30}}$$ (converges)

(b) $\displaystyle\int_{-1}^{1} \frac{1}{x^2}\,dx$ (improper b/c $y = \frac{1}{x}$ is undefined @ $x=0$, which lies in $[-1,1]$)

$$= \lim_{t\to 0^-} \int_{-1}^{t} x^{-2}\,dx + \lim_{t\to 0^+} \int_t^1 x^{-2}\,dx$$

$$= \lim_{t\to 0^-} -x^{-1}\Big|_{-1}^{t} + \lim_{t\to 0^+} -x\Big|_t^1$$

$$= \lim_{t\to 0^-}\left[\frac{-1}{t} - 1\right] + \lim_{t\to 0^+}\left[-1 + \frac{1}{t}\right]$$

$$= \infty - 1 + -1 + \infty = \boxed{\infty}$$ (diverges)

3. $y = \frac{1}{3}(x^2+2)^{3/2}$

$$\frac{dy}{dx} = x\sqrt{x^2+2}$$

$$s = \int_0^3 \sqrt{1 + \left[x\sqrt{x^2+2}\right]^2}\, dx$$

$$= \int_0^3 \sqrt{1 + x^2(x^2+2)}\, dx$$

$$= \int_0^3 \sqrt{1 + x^4 + 2x^2}\, dx$$

$$= \int_0^3 \cancel{\sqrt{(x^2+1)^{\cancel{2}}}}\, dx$$

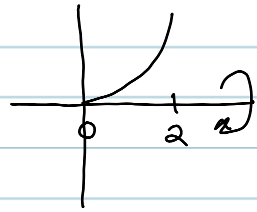
$$= \int_0^3 x^2 + 1\, dx$$

$$= \frac{x^3}{3} + x \Big|_0^3$$

$$= \frac{27}{3} + 3 + 0$$

$$= \boxed{12} \text{ units}$$

4. $y = x^3$




$$\frac{dy}{dx} = 3x^2$$

$r(x) = y = x^3$

$$S = 2\pi \int_0^2 x^3 \sqrt{1 + (3x^2)^2}\, dx$$

$$= 2\pi \int_0^2 x^3 \sqrt{1 + 9x^4}\, dx$$

$u = 1 + 9x^4$
$du = 36x^3 dx$
$dx = \frac{du}{36x^3}$

$$= 2\pi \int_{x=0}^{x=2} \cancel{x^3}\, u^{1/2} \frac{du}{36\cancel{x^3}}$$

$$= \frac{\pi}{18} \int_{x=0}^{x=2} u^{1/2}\, du$$

$$= \frac{\pi}{\cancel{18}_9} \cdot \frac{\cancel{2}}{3} (1 + 9x^4)^{3/2} \Big|_0^2$$

$$= \boxed{\frac{\pi}{27}\left[145^{3/2} - 1\right]} \text{ units}^2$$

5. pop nuggets $\approx \frac{4}{3}\big[0 + 4(200) + 2(3000)$
$+ 4(11{,}500) + 2(4000) + 4(250) + 0\big]$

$= \boxed{82{,}400 \text{ nuggets}}$

6. (Bonus Question)

(a) $\frac{dy}{dx} = \frac{1}{2}(1-x^2)^{-1/2}(-2x)$

$= \boxed{\frac{-x}{\sqrt{1-x^2}}}$

(b)

$$s = \int_0^t \sqrt{1 + \left(\frac{-x}{\sqrt{1-x^2}}\right)^2}\, dx$$

$$= \int_0^t \sqrt{1 + \frac{x^2}{1-x^2}}\, dx$$

$$= \int_0^t \sqrt{\frac{1-\cancel{x^2}+\cancel{x^2}}{1-x^2}}\, dx$$

$$= \int_0^t \frac{1}{\sqrt{1-x^2}}\, dx$$

$$= \arcsin(x)\Big|_0^t$$

$$= \boxed{\arcsin(t)}$$

(c) $\boxed{t = \sin\theta}$

(d) $\boxed{t = \sin(s)}$

Notice: We just showed
$s = \arcsin(t)$
&
$t = \sin(s)$
So, arcsin(t) really does give arc length & is the inverse of sine.