Lab 3: More on Power Series

Names:

Goal

The goal of this lab is to take a more in-depth look at power series.

Directions

In groups of 2–4, answer each of the following questions in the space provided. You only need to turn in one lab per group (make sure you put everyone's name on this sheet). The lab is due on **Wed, May 6** and is worth 10 points.

Exercises

Alright, let's do some exploring.

1. Sometimes (not always), we can use geometric series to help us determine a power series for a particular function. For example, in class, we were able to determine a power series (centered at $x = 0$) for $f(x) = \arctan x$. Let's remind ourselves how we did that. First, we noticed that

$$\frac{d}{dx}[\arctan x] = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)}.$$

The last expression on the right looks like the sum of a geometric series with $a = 1$ and $r = -x^2$. This implies that

$$\frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n}$$

as long as $|-x^2| < 1$. Simplifying the last inequality, we see that we have the above power series for $x \in (-1, 1)$. (Note that we don't need to check endpoints for power series that arise from geometric series.) However, we want the power series for $f(x) = \arctan x$, not its derivative. So, we must integrate:

$$\begin{aligned} f(x) &= \arctan x \\ &= \int \frac{1}{1+x^2}\, dx \\ &= \int \sum_{n=0}^{\infty}(-1)^n x^{2n}\, dx \\ &= \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1} + C. \end{aligned}$$

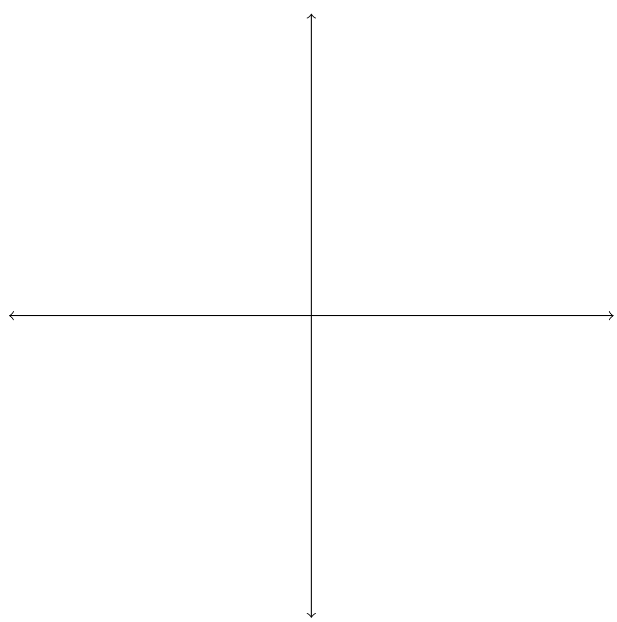
(a) Using the fact that $\arctan 0 = 0$, find the constant C in the expression above.

(b) Therefore, we can conclude that $f(x) = \arctan x$ is equal to what power series and has what radius of convergence?

(c) It turns out that we get convergence of the power series to $f(x) = \arctan x$ at both endpoints of the interval of convergence. If this is true, what is the interval of convergence of the power series for $f(x) = \arctan x$?

(d) If $s_n(x)$ denotes the sum of the first n terms of the power series for $\arctan x$, find $s_4(x)$ and $s_5(x)$. (Hint: $s_4(x)$ is of degree 7, not 4.)

(e) Graph $f(x) = \arctan x$, $y = s_4(x)$, and $y = s_5(x)$ on your calculator and then make a rough sketch below, being sure to label which is which.



(f) Describe how $s_n(x)$ and $f(x)$ are related on their interval of convergence as $n \to \infty$.

(g) Using the power series that we obtained in part (b), find the sum of the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots.$$

(Hint: what x-value could we plug into the power series to obtain the above series? What is arctangent of this x-value?)

(h) Using your answer in part (g), determine the sum of the series

$$4\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}.^*$$

2. Recall that an elementary function is any function that can be written as a finite combination (sum, difference, product, quotient, and composition) of polynomials, rational functions, power functions (including roots), exponential functions, log function, trig functions, and inverse trig functions. Back in Section 8.5, we briefly discussed the fact that some elementary functions do not have elementary antiderivatives. This means that we cannot immediately appeal to the Fundamental Theorem of Calculus to calculate some innocent looking definite integrals. Here is an example:

$$\int_0^1 e^{x^2}\, dx.$$

It turns out that $y = e^{x^2}$ does not have an elementary antiderivative. Let's see if we can use a power series to evaluate this integral.

(a) Consider the power series $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. Show that this power series satisfies the following (differential) equation.

$$f'(x) = f(x)$$

*This formula is know as the Leibniz formula for π. This beautiful formula can be used to approximate the digits of π, however, it is horribly inefficient. Calculating π to 10 correct decimal places using the Leibniz formula requires over 10,000,000,000 mathematical operations, and will take longer for most computers to calculate than calculating π to millions of digits using more efficient methods.

(b) Notice that $y = e^x$ also satisfies the above equation. This implies that e^x and the power series must differ by a constant (by Corollary 7 on page 218). Argue that this constant must be 0 and conclude that the power series in (a) is actually equal to e^x.[†] (Hint: what do you get if you plug in 0 into both functions?)

(c) Now, determine a power series for $y = e^{x^2}$ by substituting in x^2 in place of x in the power series for $y = e^x$ (think function composition).

(d) Next, evaluate $\int e^{x^2}\, dx$ in terms of a power series.

(e) Using your answer in part (d), find an infinite series solution to $\int_0^1 e^{x^2}\, dx$.

(f) Approximate $\int_0^1 e^{x^2}\, dx$ by adding up the first 5 terms of the series in part (e).

[†] It turns out that the interval of convergence here is $(-\infty, \infty)$. Also, note that we have just completed Exercise 35 from Section 12.9.