Section 7.7: Hyperbolic Functions

Goal

In this section, we will introduce the *hyperbolic (trig) functions*, study their various properties, and most importantly, see how we can use the inverse hyperbolic functions to integrate a few more functions.

The hyperbolic functions

Definition 1. We define the *hyperbolic functions* as follows.

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \text{csch } x = \frac{1}{\sinh x}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \text{sech } x = \frac{1}{\cosh x}$$

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{\cosh x}{\sinh x}$$

Note 2.

(1) We pronounce sinh, cosh, and tanh as ____________, ____________, and ____________, respectively.

(2) The trig terminology and notation stem from the fact that these functions have very similar properties to the ordinary trig functions. (There is an occasional absense or addition of a ____________.)

Here are some identities involving the hyperbolic functions.

Theorem 3.

$$\sinh(-x) = -\sinh x \qquad \cosh(-x) = \cosh x$$

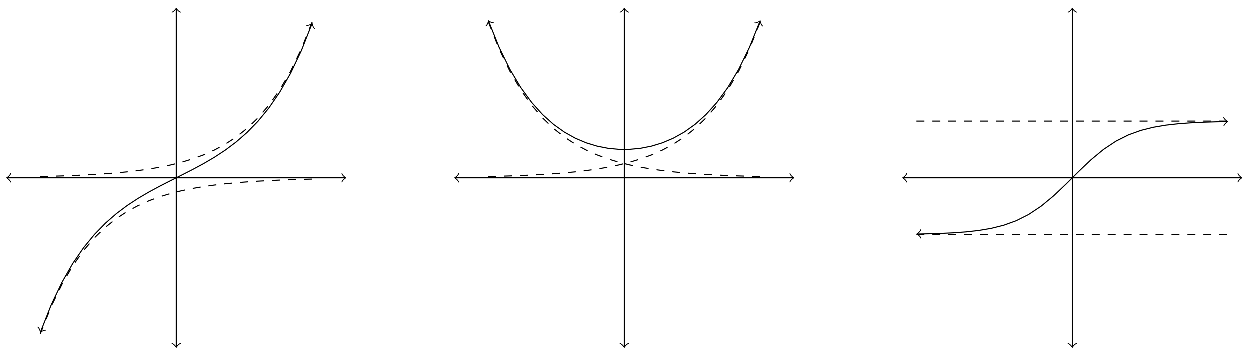
$$\cosh^2 x - \sinh^2 x = 1 \qquad 1 - \tanh^2 x = \text{sech }^2 x$$

Proof. Let's prove the third identity. The proofs of the remaining ones are similar.

$$\cosh^2 x - \sinh^2 x =$$

□

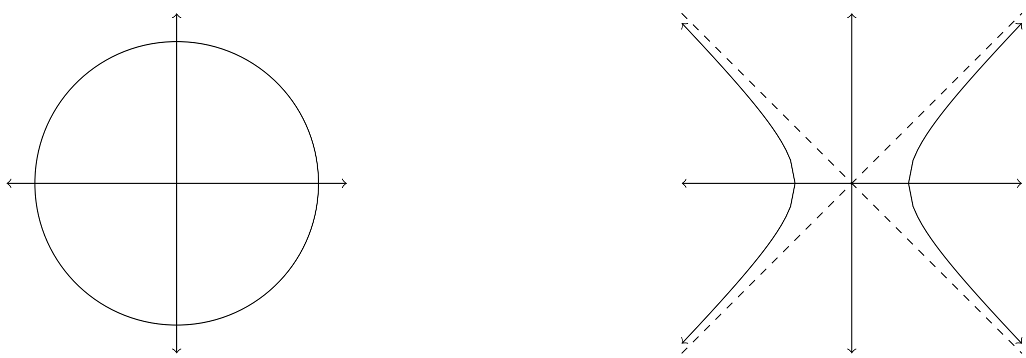
Here are the graphs of $y = \sinh x$, $y = \cosh x$, and $y = \tanh x$.



(a) $y = \sinh x$ (b) $y = \cosh x$ (c) $y = \tanh x$

Where are these asymptotes coming from?

Important Note 4. The ordinary trig functions parametrize the unit circle $x^2 + y^2 = 1$. The hyperbolic (trig) function parametrize the unit hyperbola $x^2 - y^2 = 1$. (Now, you see the reason for the differences in minus signs.)



(a) $x^2 + y^2 = 1$ (b) $x^2 - y^2 = 1$

Since the hyperbolic functions are defined in terms of e^x and e^{-x} and these functions are differentiable, the 6 hyperbolic functions are also differentiable.

Theorem 5.

$$\frac{d}{dx}[\sinh x] = \cosh x \qquad \frac{d}{dx}[\operatorname{csch} x] = -\operatorname{csch} x \coth x$$

$$\frac{d}{dx}[\cosh x] = \sinh x \qquad \frac{d}{dx}[\operatorname{sech} x] = -\operatorname{sech} x \tanh x$$

$$\frac{d}{dx}[\tanh x] = \operatorname{sech}^2 x \qquad \frac{d}{dx}[\coth x] = -\operatorname{csch}^2 x$$

Example 6. Differentiate.

(a) $f(x) = \cosh(3x - 2)$

(b) $y = (\tanh x^2)^3$

Since we know the formulas for the derivatives of the hyperbolic functions, we also get the following integration formulas.

Theorem 7.

$$\int \sinh x \ dx = \qquad \int \cosh x \ dx =$$

$$\int \operatorname{sech}^2 x \ dx = \qquad \int \operatorname{sech} x \tanh x \ dx =$$

Example 8. Integrate.

(a) $\int \sinh x \cosh x \ dx$

(b) $\int \frac{\cosh \sqrt{x}}{\sqrt{x}}$

Inverse hyperbolic functions

As with the ordinary trig functions, we can restrict the domain (if necessary) to define the inverse hyperbolic functions.

Definition 9. (See page 466 for a detailed description of these functions and pictures of their graphs.)

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right) \text{ for } x \in \mathbb{R}$$

$$\cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right) \text{ for } x \geq 1$$

$$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \text{ for } -1 < x < 1$$

Since each of the hyperbolic functions are differentiable, so are their (partial) inverses.

Theorem 10.

$$\frac{d}{dx}\left[\sinh^{-1} x\right] = \frac{1}{\sqrt{1+x^2}}$$

$$\frac{d}{dx}\left[\cosh^{-1} x\right] = \frac{1}{\sqrt{x^2-1}}$$

$$\frac{d}{dx}\left[\tanh^{-1} x\right] = \frac{1}{1-x^2}$$

How would we go about proving each of these formulas?

Example 11. Differentiate $f(x) = \ln\left(\tanh^{-1} x\right)$.

One of our main motivations for introducing hyperbolic functions was so that we could add a few more tools to our integration tool box.

Theorem 12.

$$\int \frac{1}{\sqrt{x^2+1}}\,dx =$$

$$\int \frac{1}{\sqrt{x^2-1}}\,dx =$$

$$\int \frac{1}{1-x^2}\,dx =$$

Example 13. Integrate $\displaystyle\int_0^1 \frac{x}{\sqrt{1+x^4}}\,dx$.