Section 8.1: Integration by Parts

Goal

In this section, we will introduce a technique of integration called *integration by parts*.

Motivation

Every time we have a formula for the derivative of a function, we have a corresponding integration formula. For example, we know the following derivative formula:

$$\frac{d}{dx}[\arctan x] = \underline{\qquad}.$$

This implies that we have the following corresponding integration formula:

$$\int \frac{1}{1+x^2}\, dx = \underline{\qquad\qquad}.$$

Furthermore, you may not have noticed, but every differentiation rule has a corresponding integration rule. For example, the chain rule for derivatives corresponds to u-substitution for integrals. In this section, we will see that *integration by parts* is the integration rule that corresponds to the product rule for derivatives. Integration by parts will be most useful for integrating products of functions, but as we will see, it can be used in other situations, as well.

Integration by parts

The formula for *integration by parts* is as follows.

$$\boxed{\int u\, dv = uv - \int v\, du}$$

Let's see if we can derive this formula. First, let $u(x)$ and $v(x)$ be differentiable functions of x. Recall that

$$u'(x) = \frac{du}{dx} \quad \text{and} \quad v'(x) = \frac{dv}{dx}.$$

Switching to differential form, we see that

$$du = \underline{\qquad} \quad \text{and} \quad dv = \underline{\qquad}.$$

Next, by the product rule, we have

$$\frac{d}{dx}[uv] = \underline{\qquad\quad}.$$

If we solve for uv' in the equation above, we obtain

$$uv' = \underline{\qquad\qquad}.$$

Now, if we integrate (with respect to x) both sides of the equation above, we see that

$$\int uv' \, dx = \underline{\hspace{4cm}}.$$

Lastly, if we replace things with the appropriate differentials and simplify the first integral on the right, we obtain the desired formula:

$$\int u \, dv = uv - \int v \, du.$$

Important Note 1. To use integration by parts, we need to identify

(i) u;

(ii) dv (it must be something we can integrate).

Then we must find

(iii) du (by differentiation);

(iv) v (by integration).

Note that the formula for integration by parts is what one would expect if we are dealing with a definite integral:

$$\boxed{\int_a^b u \, dv = \underline{\hspace{3cm}}.}$$

Examples

Let's do some examples.

Example 2. Integrate each of the following.

(a) $\displaystyle\int xe^{-x} \, dx$

(b) $\int x^2 \sin x \; dx$

(c) $\int \ln x \; dx$

Comments

As time goes on, our proficiency at picking the correct u and dv will increase. Here is a list of "suggestions" for common integrals using integration by parts.

1. For
$$\int x^n e^{ax}\ dx, \quad \int x^n \sin ax\ dx, \quad \int x^n \cos ax\ dx$$
let $u =$ __ and $dv =$ ______, ________, or ________.

2. For
$$\int x^n \ln x\ dx, \quad \int x^n \arcsin ax\ dx, \quad \int x^n \arctan ax\ dx$$
let $u =$ ___, ________, or ________ and $dv =$ _____.

3. For
$$\int e^{ax} \sin bx\ dx, \quad \int e^{ax} \cos bx\ dx$$
let $u =$ _____ or _____ and $dv =$ _____.

Another example

Here is one more example.

Example 3. Integrate: $\int e^x \cos 2x\ dx$