Section 9.1: Arc Length
Section 9.2: Area of a Surface of Revolution

Goal

In these two sections, we will learn how integrals can be used to find arc length and surface area.

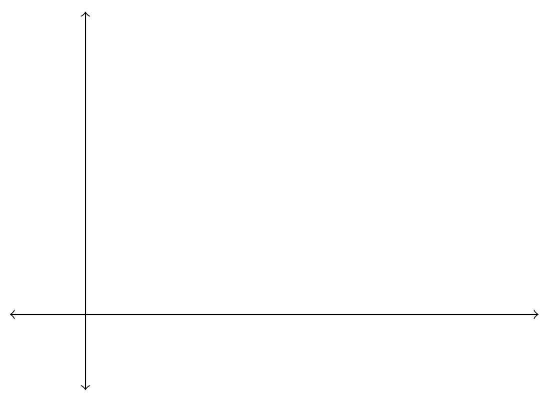
Arc length

Suppose that f is a "smooth" function (i.e., f' exists and is continuous, so that there are no sharp turns or vertical tangents on f) on a closed interval $[a, b]$.

How could we approximate the arc length using things we know how to do?

One possible answer is to partition $[a, b]$ into equal width subintervals (as we did when we approximated area). Between each pair of adjacent points, form a line segment.

Here's the picture:



In this case,

$$\text{arc length} \approx \text{the sum of the lengths of the line segments} = \sum_{i=1}^{n} d(x_{i-1}, x_i).$$

But what is each $d(x_{i-1}, x_i)$ equal to?

$$d(x_{i-1}, x_i) =$$

We have just shown that

$$\text{arc length} \approx \sum_{i=1}^{n} \sqrt{1 + [f'(c_i)]^2}\ \Delta x.$$

Well, how do you think we can get the exact value of the arc length?

$$s = \text{arc length} = \underline{\hspace{6cm}}.$$

Therefore, the arc length of the smooth curve $y = f(x)$ over the interval $[a, b]$ is given by

$$\boxed{s = \int_a^b \sqrt{1 + [f'(x)]^2}\ dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx.}$$

Sometimes $\sqrt{1 + [f'(x)]^2}\ dx$ is denoted by ds, so that

$$s = \int_a^b ds.$$

Examples

Let's do a couple of examples.

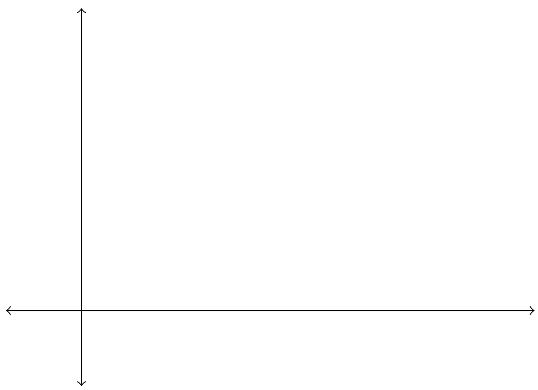
Example 1. Find the length of the curve $y = 2x^{3/2}$ over the interval $[0, 1]$.

Example 2. Prove that the circumference of the unit circle is 2π.

Surface area

Again, suppose that f is a "smooth" function on the interval $[a, b]$. We want to be able to find the surface area of the solid of revolution obtained by revolving $y = f(x)$ around the x-axis or y-axis. As usual, we first approximate and then take the limit.

Here's the picture:



The surface area of one of the approximating "slices" is equal to $2\pi r(c_i)\sqrt{1 + [f(c_i)]^2}\ \Delta x$. Adding up all of the "slices" and then taking the limit, we obtain

$$S = \text{surface area} = \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi r(c_i)\sqrt{1 + [f(c_i)]^2}\ \Delta x.$$

Therefore, the surface area of the solid of revolution obtained by revolving the smooth curve $y = f(x)$ over the interval $[a, b]$ about the x-axis (respectively, y-axis) is given by

$$\boxed{S = 2\pi \int_a^b r(x)\sqrt{1 + [f'(x)]^2}\ dx = 2\pi \int_a^b r(x)\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx,}$$

where $r(x) = f(x)$ (respectively, $r(x) = x$).

Alternatively, we may write

$$S = 2\pi \int_a^b r(x)\ ds.$$

More examples

Now, let's do a couple of surface area examples.

Example 3. Find surface area of the solid obtained by revolving the graph of $y = x^3$ on the interval $[0, 2]$ about the x-axis.

Example 4. Find surface area of the solid obtained by revolving the graph of $f(x) = \dfrac{x^5}{5} + \dfrac{1}{12x^3}$ on the interval $[1, 2]$ about the y-axis.