# Lab 5: Comparison Tests for Series

**Names:**

## Goal

The goal of this lab is to introduce you to the Direct Comparison Test and the Limit Comparison Test, which are very useful for testing convergence of a series by comparing it to a similar series where we know whether the comparable series converges or diverges.

## Directions

In groups of 2–4, answer each of the following questions in the space provided. You only need to turn in one lab per group (make sure you put everyone's name on this sheet). The lab is due on **Friday, 4.30** and is worth 10 points.

## Initial discussion

Suppose we have two series $\sum a_n$ and $\sum b_n$ of positive terms (i.e., each $a_n$ and $b_n$ are positive), where for all $n$,

$$a_n \leq b_n.$$

What can you conclude (if anything) when:

1. $\sum a_n$ diverges?

2. $\sum a_n$ converges?

3. $\sum b_n$ diverges?

4. $\sum b_n$ converges?

This leads us to the Direct Comparison Test.

# The Direct Comparison Test

**Theorem 1** (Direct Comparison Test)**.** If $\sum a_n$ and $\sum b_n$ are series with positive terms and $a_n$ ____ $b_n$ for all $n$ sufficiently large, then

(i) If $\sum a_n$ diverges, then $\sum b_n$ ______________.

(ii) If $\sum b_n$ converges, then $\sum a_n$ ______________.

**Important Note 2.** If larger series ______________, then we cannot make any conclusions about the smaller series. Similarly, if the smaller series ______________, then we cannot make any conclusions about the larger series.

**Example 3.** Determine whether each of the following series converges or diverges. (If a series converges, you do *not* need to determine its sum.)

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3}$

**Solution:** This series resembles a $p$-series with $p = 2$. So, let's compare it with $\sum_{n=1}^{\infty} \frac{1}{n^2}$, which we know ______________ (since $p > 1$). In order for this to be a useful comparison, we want $\sum_{n=1}^{\infty} \frac{1}{n^2}$ to be the larger of the two series, but is it?

Well, first observe that $2n^2 + 4n + 3 \geq n^2$, which implies

$$\frac{1}{2n^2+4n+3} \;\_\_\_\; \frac{1}{n^2}.$$

By the Direct Comparison Test, it must be the case that $\sum \frac{1}{2n^2+4n+3}$ ______________ since the larger series ______________. But what about the 5?

Since $\sum \frac{1}{2n^2+4n+3}$ ______________, it must be the case that $5 \sum \frac{1}{2n^2+4n+3} = \sum \frac{5}{2n^2+4n+3}$ also ______________.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$ (Hint: compare with the Harmonic Series for $n \geq 3$. Why?)

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{3^n+1}$ (Hint: compare with the geometric series $\sum \frac{2}{3}\left(\frac{2}{3}\right)^{n-1}$.)

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{3^n-1}$

This series looks similar to the series in (c), except we are subtracting 1 instead of adding 1. We were able to use Direct Comparison in (c) because the denominator in the given series is ____________ than series we compared it to. In turn, this made the given series ____________ than the convergent series we compared it to.

However, for this series, the denominator of the given series is ____________ than the series we'd like to compare it to. In this case, Direct Comparison will not work. We need another method.

## The Limit Comparison Test

**Theorem 4** (Limit Comparison Test)**.** If $\sum a_n$ and $\sum b_n$ are series with positive terms and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \text{finite positive number},$$

then either both series converge or both diverge.

We won't prove this theorem, but let's see if we can get some intuition about it. If $\lim_{n\to\infty} \frac{a_n}{b_n}$ is a finite positive number, then the sequences $\{a_n\}$ and $\{b_n\}$ are decreasing/increasing at relatively the same rate. If one of the sequences "shrinks" fast enough, then the other does, too.

**Important Note 5.**

1. If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$, then the Limit Comparison Test tells us nothing.
2. The order of $a_n$ and $b_n$ in the limit does not matter, although the answer for the limit may be different (but won't change whether it is finite and positive).

 

**Example 6.** Determine whether each of the following series converges or diverges. (If a series converges, you do *not* need to determine its sum.)

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{3^n - 1}$ (Hint: use Limit Comparison Test with the geometric series $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$.)

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2 + 3n}{\sqrt{5 + n^5}}$ (Hint: use Limit Comparison Test with the $p$-series $\sum \frac{1}{n^{1/2}}$. Why?)

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n + \ln n}$ (Hint: use Limit Comparison Test with $\sum \frac{1}{n}$. Why?)

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