Section 12.1: Sequences

Goal

In this section, we introduce sequences, including some terminology, and look at several examples.

Definition of a sequence and basic examples

Definition 1. A *sequence* is a list (usually infinite) of objects (usually numbers, but does not have to be) that has a specified order. More specifically, a sequence is a function with domain $\mathbb{N}$.
We usually denote the *terms* of a sequence with subscripts:

$$a_1, \underbrace{a_2}_{\text{2nd}}, a_3, \ldots, \underbrace{a_n}_{n\text{th}}, \underbrace{a_{n+1}}_{(n+1)\text{th}}, \ldots$$

We denote the entire sequence by $\{a_n\}_{n=1}^{\infty}$ (or possibly $(a_n)_{n=1}^{\infty}$) or simply $\{a_n\}$ (respectively, (a_n)).

Example 2.

(a) $\{a_n\} = \{1, 2, 3, 4, \ldots\}$. In this case, $a_n =$____________. Also:

$a_3 =$____________

$a_{16} =$____________

(b) $\{b_k\} = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$. In this case, $b_k =$____________. Also:

$b_4 =$____________

$b_7 =$____________

(c) $\{c_n\} = \{\frac{3}{5}, \frac{-4}{25}, \frac{5}{125}, \frac{-6}{625}, \frac{7}{3125} \ldots\}$. In this case, $c_n =$____________. Also:

$c_3 =$____________

$c_{12} =$____________

(d) $\{a_i\} = \{1, 1, 1, \ldots\}$. In this case, $a_i =$____________. This is an example of a ____________. Also:

$a_3 =$____________

$a_{100} =$____________

(e) $\{p_n\} = \{2, 3, 5, 7, 11, 13, 17, \ldots\}$ (sequence of prime numbers). In this case, there is no formula for p_n. However, we can find:

$p_3 =$____________

$p_7 =$____________

(f) $\{f_n\} = \{1, 1, 2, 3, 5, 8, 13, \ldots\}$ (called the *Fibonacci sequence*). This is an example of a *recursive sequence*. In this case, $f_1 =$____________, $f_2 =$____________, and $f_n =$____________for $n \geq$ ____________. Also:

$f_6 =$______________

$f_{11} =$______________

Note 3. Sometimes we will begin a sequence at a different index other than 1.

Example 4.

(a) Consider the sequence $\{c_n\}$ in Example 2(c). What is $\{c_n\}_{n=3}^{\infty}$?

(b) Consider the sequence $\{b_k\}$ in Example 2(b) and compare with $\{b'_k\}_{k=0}^{\infty}$, where $b'_k = \frac{1}{k+2}$.

We can draw "graphs" of sequences, where the x-axis is $\mathbb{N}$ and the y-axis is the set of values that the sequence takes (for us, the y-axis will usually be $\mathbb{R}$).

Example 5. Draw a graph of the sequence $\left\{\frac{(-1)^{n+1}}{n}\right\}$.

Limits of sequences

Definition 6. A sequence $\{a_n\}$ has the *limit* L and we write

$$\lim_{n\to\infty} a_n = L$$

if we can make the terms of $\{a_n\}$ as close to L as we like by taking n sufficiently large. If such a L exists, we say that the sequence ______________. Otherwise, we say that the sequence ______________.

The Picture:

Let's play with the applet located at http://calculusapplets.com/sequence.html.

Theorem 7. *If* $\lim_{n\to\infty} f(x) = L$ *and* $f(n) = a_n$, *then* $\lim_{n\to\infty} a_n = L$, *as well.*

Important Note 8. What this means is that we get to use all of our previous limit weapons (limit laws, Squeeze Theorem, L'Hospital's Rule, etc.).

Here is another weapon.

Theorem 9. *If* $\lim_{n\to\infty} |a_n| = 0$, *then* $\lim_{n\to\infty} a_n =$_____________.

Example 10. Converge or diverge? If the sequence converges, find its limit.

(a) $\left\{\frac{1}{n}\right\}$

(b) $\left\{\frac{3n^2+2n+2}{1-n^2}\right\}$

(c) $\{(-1)^n\}$

(d) $\left\{\frac{(-1)^n}{n^2+1}\right\}$

(e) $\left\{\frac{\ln n}{n}\right\}$

(f) $\{\sin(\pi/n)\}$

(g) $\left\{\frac{n!}{n^n}\right\}$ (Hint: use Squeeze Theorem)

More terminology

Definition 11.

1. $\{a_n\}$ is *(strictly) increasing* if _____________for all $n \geq 1$.
2. $\{a_n\}$ is *(strictly) decreasing* if _____________for all $n \geq 1$.
3. $\{a_n\}$ is *monotonic* if it is either increasing or decreasing.
4. $\{a_n\}$ is *bounded above* (respectively, *below*) if there exists $M \in \mathbb{R}$ such that _____________(respectively, _____________).

Theorem 12 (Monotonic Sequence Theorem)**.** *Every bounded (above and below) monotonic sequence is convergent.*