

## Section 9.1: Arc Length

## Section 9.2: Area of a Surface of Revolution

### Goal

In these two sections, we will learn how integrals can be used to find arc length and surface area.

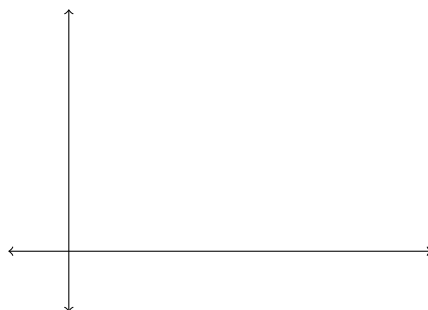
### Arc length

Suppose that  $f$  is a “smooth” function (i.e.,  $f'$  exists and is continuous, so that there are no sharp turns or vertical tangents on  $f$ ) on a closed interval  $[a, b]$ .

How could we approximate the arc length using things we know how to do?

One possible answer is to partition  $[a, b]$  into equal width subintervals (as we did when we approximated area). Between each pair of adjacent points, form a line segment.

Here's the picture:



In this case,

$$\text{arc length} \approx \text{the sum of the lengths of the line segments} = \sum_{i=1}^n d(x_{i-1}, x_i).$$

But what is each  $d(x_{i-1}, x_i)$  equal to?

$$d(x_{i-1}, x_i) =$$

We have just shown that

$$\text{arc length} \approx \sum_{i=1}^n \sqrt{1 + [f'(c_i)]^2} \Delta x.$$

Well, how do you think we can get the exact value of the arc length?

$$s = \text{arc length} = \underline{\hspace{10cm}}.$$

Therefore, the arc length of the smooth curve  $y = f(x)$  over the interval  $[a, b]$  is given by

$$s = \int_a^b \sqrt{1 + [f'(x)]^2} dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx.$$

Sometimes  $\sqrt{1 + [f'(x)]^2} dx$  is denoted by  $ds$ , so that

$$s = \int_a^b ds.$$

## Examples

Let's do a couple of examples.

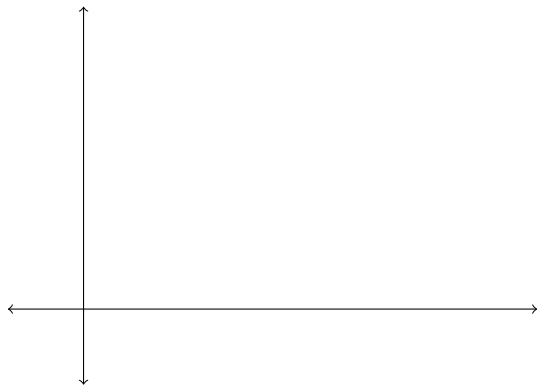
**Example 1.** Find the length of the curve  $y = 2x^{3/2}$  over the interval  $[0, 1]$ .

**Example 2.** Prove that the circumference of the unit circle is  $2\pi$ .

## Surface area

Again, suppose that  $f$  is a “smooth” function on the interval  $[a, b]$ . We want to be able to find the surface area of the solid of revolution obtained by revolving  $y = f(x)$  around the  $x$ -axis or  $y$ -axis. As usual, we first approximate and then take the limit.

Here’s the picture:



The surface area of one of the approximating “slices” is equal to  $2\pi r(c_i)\sqrt{1 + [f'(c_i)]^2} \Delta x$ . Adding up all of the “slices” and then taking the limit, we obtain

$$S = \text{surface area} = \lim_{n \rightarrow \infty} \sum_{i=1}^n 2\pi r(c_i)\sqrt{1 + [f'(c_i)]^2} \Delta x.$$

Therefore, the surface area of the solid of revolution obtained by revolving the smooth curve  $y = f(x)$  over the interval  $[a, b]$  about the  $x$ -axis (respectively,  $y$ -axis) is given by

$$S = 2\pi \int_a^b r(x)\sqrt{1 + [f'(x)]^2} dx = 2\pi \int_a^b r(x)\sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx,$$

where  $r(x) = f(x)$  (respectively,  $r(x) = x$ ).

Alternatively, we may write

$$S = 2\pi \int_a^b r(x) ds.$$

## More examples

Now, let’s do a couple of surface area examples.

**Example 3.** Find surface area of the solid obtained by revolving the graph of  $y = x^3$  on the interval  $[0, 2]$  about the  $x$ -axis.

**Example 4.** Find surface area of the solid obtained by revolving the graph of  $f(x) = \frac{x^5}{5} + \frac{1}{12x^3}$  on the interval  $[1, 2]$  about the  $y$ -axis.