This research statement is organized as follows. Section 1 provides a quick summary of my research interests. Then in Section 2, I provide a more in-depth description of three current research projects: (i) conjugacy and reducibility in Coxeter groups, (ii) diagram algebras and Kazhdan–Lusztig theory, and (iii) mathematics education and IBL. In Section 3, I summarize recent and ongoing research with undergraduates. Lastly, for readers desiring a more detailed description of (i) and (ii) above, Sections 4 and 5 elaborate on my current results and plans for future research in these areas. **If a reader is only interested in an overview of my research interests, they can safely skip reading Sections 4 and 5**.

1 Introduction

My primary research interests are in the interplay between combinatorics and algebraic structures. More specifically, I study the combinatorics of Coxeter groups and their associated Hecke algebras, Kazhdan–Lusztig theory, generalized Temperley–Lieb algebras, diagram algebras, and heaps of pieces. By employing combinatorial tools such as diagram algebras and heaps of pieces, one can gain insight into algebraic structures associated to Coxeter groups, and, conversely, the corresponding structure theory can often lead to surprising combinatorial results. The combinatorial nature of my research naturally lends itself to collaboration with advanced undergraduate students, and I strive to incorporate undergraduates in my research whenever possible.

I am also passionate about undergraduate mathematics education and recently my research interests have included topics in this area. In particular, I am studying the effectiveness of a collaborative approach to inquiry-based learning (IBL) in proof-based courses. Furthermore, I am interested in the use of technology, such as Sage, wikis, and other Web2.0 technology, to aid in the learning of mathematics. Sage is a free open-source mathematics software system whose mission is to create a viable alternative to Maple, Mathematica and Matlab.

2 Overview of research program

2.1 Conjugacy and reducibility in Coxeter groups

This section describes ongoing joint work with R.M. Green (University of Colorado) and M. Macauley (Clemson University).

Matsumoto's theorem states that any two reduced expressions of the same element in a Coxeter group differ by a sequence of braid relations, which elegantly solves the word problem. Cyclically shifting a reduced expression is conjugation by the initial letter. Now, consider the following question.

> *Do two cyclically reduced expressions of conjugate elements differ by a sequence of braid relations and cyclic shifts?*

This question is in a sense a "cyclic version" of Matsumoto's theorem, and while the answer is "no," it seems to "usually be true." Characterizing when the cyclic version of Matsumoto's theorem holds is a problem that is rich in combinatorics, and resolving it would solve the conjugacy problem for Coxeter groups *combinatorially*. This question arose in an attempt to cast several recent results on conjugacy in a more natural setting. Recently, D. Speyer proved that in infinite irreducible Coxeter groups, powers of Coxeter elements are reduced [31], and then H. Eriksson and K. Eriksson used that result to give a combinatorial solution of the conjugacy problem for Coxeter elements [6], finishing a problem first posed in H. Eriksson's 1994 PhD thesis [5]. An unstated corollary of this work is an affirmative answer to the cyclic version of Matsumoto's theorem for Coxeter elements. That is, for Coxeter elements, any two reduced expressions up to conjugacy are equivalent under the equivalence generated by braid relations *and* cyclic shifts. We are certain that these results are special cases of a broader mathematical framework that has never been studied directly. In particular, Coxeter elements have the property that every generator appears precisely once, but this is much stronger than what was needed for the aforementioned results. One of our goals is to extend current results to a more unifying general statement for arbitrary conjugacy classes. The first generalization of Coxeter elements for which this combinatorics applies are the *cyclically fully commutative* (CFC) elements, introduced by Green, Macauley, and myself in recent work. Loosely speaking, these are the elements whose reduced expressions under the "braid + cyclic shift" equivalence all lie in the same equivalence class, and they are rich in combinatorics. The most general case would be to extend these results from the CFC to the *cyclically reduced* elements.

Preliminary investigations have yielded a wealth of new research problems and a clear path of how to attack the question posed above. I have coauthored one paper [3] in this area. Our main result in [3] extends Speyer's result to the CFC elements for a large class of groups that contains all Weyl and simply-laced Coxeter groups (see Theorem 1 in Section 4.3). Our result supports the claim that CFC elements are a natural generalization of Coxeter elements, and it is a foundation for our work on the cyclic version of Matsumoto's theorem and the conjugacy problem.

2.2 Diagram algebras and Kazhdan–Lusztig theory

This section describes results related to my PhD thesis [8], which has since produced three papers [9, 10, 11]. The first paper appears in print, the second is currently under review, and the third is awaiting acceptance of the second paper. I outline further research in this area in Section 5.2.

The (type A) Temperley–Lieb algebra $\mathrm{TL}(A)$, invented by H.N.V. Temperley and E.H. Lieb in 1971 [33], is a finite dimensional associative algebra, which arose in statistical mechanics. R. Penrose and L.H. Kauffman showed that this algebra can be realized as a particular *diagram algebra* [26, 30], which is a type of associative algebra with a basis given by certain diagrams in which the multiplication rule is given by applying local combinatorial rules to the diagrams. In 1987, V.F.R. Jones showed that $\mathrm{TL}(A)$ occurs naturally as a quotient of the type A Hecke algebra, $\mathcal{H}(A)$, whose underlying group is the symmetric group [25]. Jones introduced a Markov trace on $\mathcal{H}(A)$ that is degenerate (the trace is the matrix trace of a transfer matrix algebra), but its radical is an ideal of $\mathcal{H}(A)$, and so we obtain a generically nondegenerate trace on the quotient algebra. This quotient algebra is isomorphic to $\mathrm{TL}(A)$.

Eventually, this realization of the Temperley-Lieb algebra as a Hecke algebra quotient was generalized by J.J. Graham in [17] to the case of an arbitrary Coxeter graph Γ, which we denote by $\mathrm{TL}(\Gamma)$. Subsequently, several diagrammatic representations of these generalized Temperley–Lieb algebras have been constructed for various Coxeter systems. In a series of papers [18, 19, 20, 21], R.M. Green constructed faithful diagrammatic representations of $\mathrm{TL}(\Gamma)$, where Γ is of type B, D, or H. Martin and Saleur introduced a diagram calculus for the generalized Temperley–Lieb of type affine A [29], but faithfulness was later proved by Green and C.K. Fan in [15]. In [35], T. tom Dieck described a diagrammatic representation of the generalized Temperley–Lieb algebra of type E, which was proved to be faithful in a recent paper by Green [23]. Besides having a point of contact with physics, knot theory, and the theory of subfactors, these diagrammatic representations provide combinatorially tractable models for Kazhdan–Lusztig theory. In this vein, the main goal of my PhD thesis [8] was to construct a faithful diagrammatic representation of a generalized Temperley–Lieb algebra of type affine C, denoted by $\mathrm{TL}(\widetilde{C})$.

In [10] and [11], I construct an associative diagram algebra and prove that it is a faithful representation of $\mathrm{TL}(\widetilde{C})$. Since Coxeter groups of type $\widetilde{C}$ have an infinite number of fully commutative elements (in the sense of Stembridge), $\mathrm{TL}(\widetilde{C})$ is infinite dimensional. This is the first faithful representation of an infinite dimensional non-simply-laced generalized Temperley–Lieb algebra (in the sense of Graham). In the finite dimensional case, counting arguments are employed to prove faithfulness, but these techniques are not available in the type $\widetilde{C}$ case. Instead, we rely on my classification in [9] of the "non-cancellable" elements in Coxeter groups of types B and $\widetilde{C}$.

Our motivation for studying the non-cancellable elements stems from the fact that computation involving the monomial basis elements of the generalized Temperley–Lieb algebra of W indexed by non-cancellable elements is "well-behaved." In fact, the classification of the non-cancellable elements in [9] provides the foundation for inductive arguments used to prove the faithfulness of our diagram algebra. The classification of the non-cancellable elements in a Coxeter group of type B verifies Fan's unproved claim in [14] about the set of fully commutative elements in a Coxeter group of type B having no generator in the left or right descent set that can be left or right cancelled, respectively.

2.3 Mathematics education and IBL

For three consecutive semesters, I taught an introduction to proof course to mathematics majors at Plymouth State University. The first two iterations of the course were taught via a traditional lecture-based approach, while the third instance of the course was taught using an IBL approach with an emphasis on collaboration. When I taught an abstract algebra course with students from both styles of the introduction to proof course, anecdotal evidence suggested that the students taught via IBL were stronger proof-writers and more independent as learners. Inspired by the apparent effectiveness of IBL, I adopted this approach in my real analysis course and chose to study it with mathematics education specialist A. Hodge (University of Nebraska at Omaha). We recently submitted a short paper [12] that presents quantitative data supporting the effectiveness of a collaborative IBL approach, and qualitative data describing student perception of knowledge acquisition with regards to proof in an upper-level mathematics course.

In the spring of 2011, A. Schultz (Wellesley College) and I chose to adopt an IBL approach in our number theory courses at our respective universities. Two times during the semester, students in each class submitted proofs of 2–3 theorems to be peer reviewed by students in the other class. Each student was then responsible for typing up an anonymous and formal referee report of the submitted theorems, which were then returned to the respective students. Together with A. Hodge, we developed a pre- and post-test survey to study the impact of this form of peer review, as well as student perception of the effectiveness of IBL, in general. We are currently in the process of writing a short paper [13] in which we will relay the similarities and differences between our approaches to IBL in each course, describe the details of the peer review project, and discuss the results of the survey as it relates to peer review.

3 Undergraduate research

As mentioned in the introduction, the combinatorial nature of my research naturally lends itself to collaboration with undergraduates. Mentoring undergraduate research combines my passion for teaching with my desire to remain active in mathematics research. Many aspects of my research involve combinatorial representations that are visually appealing for students. Not only are these representations nice to look at, but they can provide insight into the underlying algebraic structure that we may not have otherwise noticed. Often problems in my research can be distilled down to asking questions about the visual representations, which provides a nice entry point for undergraduates to tackle open problems.

Before describing a few recent projects, we need a little background. In Kazhdan-Lusztig theory, it is often useful to work with elements that are either a product of commuting generators, or have a reduced expression beginning or ending with a pair of non-commuting generators. Motivated by this claim, we say that w in a Coxeter group W is *T-avoiding* if no reduced expression for w either begins or ends with a pair of non-commuting generators.

3.1 Exploration of T-avoiding elements in Coxeter groups of type F

During the 2011–2012 academic year, I am mentoring Ryan Cross, Katie Hills-Kimball, and Christie Quaranta on an original research project aimed at exploring the T-avoiding elements in Coxeter groups of type F. Preliminary results have yielded an infinite class of elements that are T-avoiding but not equal to a product of commuting generators. The students are attempting to prove that their list is exhaustive and will present their findings during at least one conference.

3.2 Classification of T-avoiding permutations in Coxeter groups of types A and B

During the 2010–2011 academic year, I mentored Joseph Cormier, Zachariah Goldenberg, Jessica Kelly, and Christopher Malbon on an original research project that classified of the T-avoiding permutations in Coxeter groups of types A and B. The students proved that in Coxeter groups of type A and B, an element is T-avoiding if and only if it is the product of commuting generators. This result is already known, but our proof is constructive and only uses elementary results. The students made the following presentations:

- *Classification of the T-avoiding permutations and generalizations to other Coxeter groups*, Undergraduate Student Poster Session, Joint Mathematics Meetings 2012, Boston, MA; January 6, 2012.
- *Classification of the T-avoiding permutations and generalizations to other Coxeter groups*, Combinatorics of Coxeter Groups Special Session, Spring 2011 Eastern Sectional Meeting of the AMS, College of the Holy Cross, Worcester, MA; April 10, 2011.
- *Classification of the T-avoiding permutations*, 2011 Hudson River Undergraduate Mathematics Conference, Skidmore College, Saratoga Springs, NY; April 16, 2011.

We are currently in the progress of writing up our results [4].

3.3 Counting generators in Temperley–Lieb diagrams of types A and B

In the spring of 2010, I mentored Sarah Otis and Leal Rivanis on a project that lies in the intersection of the research described in Sections 2.2 and 2.1. The students obtained original results concerning Temperley–Lieb diagram algebras of types A and B. In particular, we obtained a non-recursive method for enumerating the number of generators occurring in the fully commutative element that indexes a given diagram. One consequence of our results is a classification of the diagrams of the Temperley–Lieb algebras of types A and B indexed by CFC elements. The students made the following presentation:

- *Counting generators in type B Temperley–Lieb diagrams*, 2010 Hudson River Undergraduate Mathematics Conference, Keene State College, Keene, NH; April 17, 2010.

4 Conjugacy and reducibility in Coxeter groups

This section elaborates on my work introduced in Section 2.1 and can be safely skipped if the reader is only interested in an overview of my research.

4.1 Preliminaries

A *Coxeter group* is a group W with a distinguished set of generating involutions S with presentation given by $\langle s_1, \ldots, s_n : (s_i s_j)^{m(s_i,s_j)} = 1\rangle$, where $m(s_i, s_j) = 1$ if and only if $i = j$. The pair (W, S) is called a *Coxeter system*, where $S = \{s_1, \ldots, s_n\}$. A Coxeter system can be encoded by a unique *Coxeter graph* Γ which has vertex set S and edges $\{s, t\}$ for each $m(s, t) \geq 3$. Moreover, each edge is labeled with the corresponding $m(s, t)$, although typically the labels of 3 are omitted because they are the most common. If Γ is connected, then (W, S) is *irreducible*. If a vertex s in Γ has degree 1, call it an *endpoint*. An endpoint s has a unique $t \in S$ for which $m(s, t) \geq 3$, and we call $m(s, t)$ the *weight* of the endpoint. If this weight is greater than 3, we say that the endpoint is *large*.

Let S^* denote the free monoid over S. A word $s_{x_1} s_{x_2} \cdots s_{x_m} \in S^*$ that is equal to $w \in W$ when considered as a group element is called an *expression* for w. If m is minimal, we say that the expression is a *reduced expression* for w, and call m the *length* of w, denoted $\ell(w)$. If every cyclic shift of a reduced expression is a reduced expression for some element in W, we say that the reduced expression is *cyclically reduced*. A group element $w \in W$ is cyclically reduced if every reduced expression for w is cyclically reduced.

For $w \in W$, we say that $s \in S$ is *initial* (respectively, *terminal*) if $\ell(sw) < \ell(w)$ (respectively, $\ell(ws) < \ell(w)$). It is well-known that if $s \in S$, then $\ell(sw) = \ell(w) \pm 1$, and so $\ell(w^k) \leq k \cdot \ell(w)$. If equality holds for all $k \in \mathbb{N}$, we say that w is *logarithmic*. For each integer $m \geq 0$ and distinct $s, t \in S$, define $\langle s,t\rangle_m = stst\cdots$ where the product on the right has m factors. The relation $\langle s,t\rangle_{m(s,t)} = \langle t,s\rangle_{m(s,t)}$ is called a *braid relation*, and additionally a *short braid relation* if $m(s,t) = 2$. The short braid relations generate an equivalence relation on S^*, and the resulting equivalence classes are called *commutation classes*. An element $w \in W$ is *fully commutative* (FC) if all of its reduced expressions lie in the same commutation class, and we denote the set of FC elements by $\mathrm{FC}(W)$. Matsumoto's theorem [16, Theorem 1.2.2] says that in a Coxeter group, any two reduced expressions for the same group element differ by a sequence of braid relations. As a consequence of Matsumoto's theorem, it is well-defined to speak of the *support* of $w \in W$ as the set of generators appearing in any reduced expression for w. If the support of w equals all of S, then we say that w has *full support*. If every connected component of the subgraph of Γ induced by the support of w describes an infinite Coxeter group, then we say that w is *torsion-free*.

4.2 Cyclic version of Matsumoto's theorem

Let W be a Coxeter group. We say that a subset $W' \subseteq W$ satisfies the *cyclic version of Matsumoto's theorem* if any two cyclically reduced expressions of conjugate group elements differ by braid relations and cyclic shifts. One only needs to look at type A_n (the symmetric group S_{n+1}) to find an example where the cyclic version of Matsumoto's theorem fails: any two simple generators are conjugate, e.g., $s_1s_2(s_1)s_2s_1 = s_2$. However, for longer words, such examples appear to be less common, and we would like to characterize exactly when they can happen, which would establish when the cyclic version of Matsumoto's theorem holds.

Matsumoto's theorem implies that if w is FC then any two reduced expressions for w differ only by *short* braid relations. The cyclic version asks when any two cyclically reduced expressions of conjugate group elements differ only by short braid relations and cyclic shifts. This problem is rich in combinatorics, and it leads to the definition of the cyclically fully commutative (CFC) elements, which can be thought of as the "cyclic analog" of FC elements.

An example of an FC (and CFC) element is a *Coxeter element*, which is a product of all the generators of S in some order. We denote the set of Coxeter elements by $\mathrm{C}(W)$. Observe that conjugating a Coxeter element $c = s_{x_1}\cdots s_{x_n}$ by s_{x_1} cyclically shifts the word to $s_{x_2}\cdots s_{x_n}s_{x_1}$. We say that two Coxeter elements $c, c' \in \mathrm{C}(W)$ are κ-equivalent if they are conjugate by a word $w = s_{x_1}\cdots s_{x_k}$ such that length is preserved after successive conjugation by $s_{x_1}, s_{x_2}, \ldots, s_{x_k}$. Though this is in general a stronger condition than just conjugacy, a recent result by H. Eriksson and K. Eriksson [6] shows that they are equivalent for Coxeter elements, thus establishing the cyclic version of Matsumoto's theorem for Coxeter elements. The proof of this rests on a recent result that Coxeter elements in infinite irreducible Coxeter groups are logarithmic. This easy-to-state result is quite non-trivial, and both known proofs constitute an entire paper each [7, 31].

4.3 Cyclically fully commutative elements

We say that an element $w \in W$ is *cyclically fully commutative* (CFC) if every cyclic shift of every reduced expression of w is a reduced expression for an FC element. The CFC elements are precisely the elements whose reduced expressions can be written in a circle without any $\langle s,t\rangle_{m(s,t)}$ subwords, where $m(s,t) > 2$. It is clear that $\mathrm{C}(W) \subseteq \mathrm{CFC}(W) \subseteq \mathrm{FC}(W)$. In 2007, M. Kleiner and A. Pelley used techniques from representation theory to prove that in an infinite irreducible simply-laced Coxeter group (i.e. all $m(s,t) \leq 3$), Coxeter elements are logarithmic [28]. Kleiner later suggested to Speyer that he seek a purely combinatorial proof of this result, which he did in 2007 [31], dropping the simply-laced condition in the process. In all of these cases, the distinguishing property of a Coxeter element, that every generator occurs precisely once, is convenient but stronger than necessary. The following theorem extends Speyer's result to the CFC elements for a large class of groups that contains all Weyl and simply-laced groups.

Theorem 1 (Boothby–Burkert–Eichwald–Ernst–Green–Macauley [3]). Let W be a Coxeter group without large odd endpoints. An element $w \in \mathrm{CFC}(W)$ is logarithmic if and only if it is torsion-free.[1]

In [32], J. Stembridge classified the Coxeter groups that contain only finitely many FC elements, which he called the *FC-finite* groups. Similarly, we define the *CFC-finite* groups to be those that contain only finitely many CFC elements. While it is clear that the every FC-finite group is also CFC-finite, we also proved that a CFC-finite group is also FC-finite.

In [3], we show that for all cases except the dihedral groups and type H, the CFC elements are precisely the Coxeter elements in the standard parabolic subgroups. It is known that the FC elements in type A are the 321-avoiding permutations [2]. A 2007 result by B.E. Tenner [34] applied to our classification of the CFC elements in CFC-finite groups tells us that the CFC elements in type A are the permutations with Boolean principal order ideals, which satisfy the following pattern avoidance condition.

Theorem 2 (Boothby–Burkert–Eichwald–Ernst–Green–Macauley [3]). An element $w \in W(A_n)$ is CFC if and only if w is 321-avoiding and 3412-avoiding.

4.4 Summary of research problems

The research problems described below all fit into the broad goal of studying reducibility and conjugacy in Coxeter groups.

Problem 1. Determine whether we can loosen the restrictions on Theorem 1. In particular, is it true that a CFC element is logarithmic if and only if it is torsion-free? It is tempting to conjecture this for purely aesthetic reasons, and it may in fact be true. However, we do not have any firm mathematical evidence.

Problem 2. Formulate and prove a cyclic version of Matsumoto's theorem. This would be a combinatorial solution to the conjugacy problem for Coxeter groups. The first step is to extend the techniques used with Coxeter elements to CFC elements.

Problem 3. Give complete combinatorial characterizations of the CFC elements and their conjugacy classes in the CFC-finite groups. In particular, one goal would be to state and prove necessary and sufficient conditions for CFC elements in a CFC-finite group to be conjugate. Another goal is to classify the CFC elements in each of the CFC-finite groups (types B, D, E, F, H, $I_2(m)$) by generalized pattern avoidance [1].

5 Diagram algebras and Kazhdan–Lusztig theory

Below, I provide more details about my research in the area of diagram algebras and its connection to Kazhdan–Lusztig theory.

5.1 Generalized Temperley–Lieb algebras

Let (W, S) be a Coxeter system with Coxeter graph Γ. The associated Hecke algebra $\mathcal{H}(\Gamma)$ is an algebra with a basis given by $\{T_w : w \in W\}$, and with relations that deform the relations of W by a parameter q. If we set q to 1, we recover the group algebra of W. In their 1979 paper [27], Kazhdan and Lusztig defined two remarkable bases $\{C_w : w \in W\}$ and $\{C'_w : w \in W\}$ for $\mathcal{H}(\Gamma)$ in terms of the natural basis.

The entries in the change of basis matrix give rise to the *Kazhdan–Lusztig polynomials* $\{P_{x,w} : x, w \in W\}$. If $x < w$ (x is a subexpression of w), then $P_{x,w}$ is a polynomial in q of degree at most $(\ell(w) - \ell(x) - 1)/2$. We let $\mu(x, w)$ denote the (integer) coefficient of $q^{(\ell(w)-\ell(x)-1)/2}$ in $P_{x,w}$. Note that $\mu(x, w)$ can only be nonzero if $x < w$ and $\ell(w) - \ell(x)$ is odd. The μ-values also appear in multiplication formulas for the Kazhdan–Lusztig basis elements $\{C'_w\}$.

[1]Note that the main result of [3] is actually a stronger statement, but for the sake of space, we stated a corollary here.

The Kazhdan–Lusztig polynomials are of great importance in algebra and geometry. They have applications to the representation theory of semisimple algebraic groups, Verma modules, algebraic geometry and topology of Schubert varieties, canonical bases, immanant inequalities, etc. Unfortunately, computing the polynomials $P_{x,w}$ efficiently quickly becomes difficult, even in finite groups of moderate size. The only obvious way to compute the $P_{x,w}$ is by means of a recurrence formula [27]:

$$P_{x,w} = q^{1-c} P_{sx,v} + q^c P_{x,v} - \sum_{sz<z} \mu(z,v) q_z^{-1/2} q_w^{1/2} P_{x,z},$$

where we define $c = 0$ if $x < sx$ and $c = 1$ otherwise. Note that the μ-values play a major role in the recursive structure of the Kazhdan–Lusztig polynomials. Computing each μ-value is not known to be any easier than computing the entire polynomial $P_{x,w}$. However, one can see from the recurrence above that computation of $P_{x,w}$ would be simplified if one could quickly compute the μ-values.

Let $\mathcal{J}(\Gamma)$ be the two-sided ideal of $\mathcal{H}(\Gamma)$ generated by the elements $\sum_{w \in \langle s,t \rangle} T_w$, where (s,t) runs over all pairs of elements of S with $3 \leq m(s,t) < \infty$, and $\langle s,t \rangle$ is the parabolic subgroup generated by s and t. We define the *generalized Temperley-Lieb algebra*, $\mathrm{TL}(\Gamma)$, to be the quotient algebra $\mathcal{H}(\Gamma)/\mathcal{J}(\Gamma)$.

One motivation behind studying these generalized Temperley–Lieb algebras is that they provide a gateway to understanding the Kazhdan–Lusztig theory of the associated Hecke algebra. Loosely speaking, $\mathrm{TL}(\Gamma)$ retains some of the relevant structure of $\mathcal{H}(\Gamma)$, yet is small enough that the computation of the leading μ-coefficients of the Kazhdan–Lusztig polynomials is often simpler.

In [24], Green and J. Losonczy show that $\mathrm{TL}(\Gamma)$ admits a canonical (or IC) basis, $\{c_w : w \in \mathrm{FC}(W)\}$. This basis is analogous to the Kazhdan–Lusztig basis for $\mathcal{H}(\Gamma)$. In addition, under some circumstances, c_w is the image of the Kazhdan–Lusztig basis element C'_w in the quotient when $w \in \mathrm{FC}(W)$.

Using the corresponding diagram algebra of $\mathrm{TL}(\Gamma)$ when Γ is of types A, B, D, or E, Green constructed a trace on $\mathcal{H}(\Gamma)$ similar to the Jones trace in type A [22]. This trace satisfies the Markov condition, which arises in the context of knot theory. The coefficient $\mu(x,w)$ appears as the coefficient of $q^{-1/2}$ in the trace of $C'_x C'_w$. Remarkably, this trace is easy to compute in the known examples if $x, w \in \mathrm{FC}(W)$, even though the problem of computing the product $C'_x C'_w$ is difficult in general.

5.2 Summary of research problems

This area of mathematics has rich combinatorial and algebraic foundations, promising extensive applications to many branches of mathematics. Below, I outline a few interesting problems for future research.

Problem 4. Using the diagrammatic representation of $\mathrm{TL}(\widetilde{C})$ defined in [10], construct a trace on $\mathcal{H}(\widetilde{C})$ and then use this trace to non-recursively compute $\mu(x,w)$ for $x, w \in \mathrm{FC}(W)$.

Problem 5. Construct a faithful diagrammatic representation of $\mathrm{TL}(F)$ and use the diagram calculus to construct a generalized Jones trace on $\mathcal{H}(F)$. Type F is the only remaining FC-finite Weyl group whose corresponding generalized Temperley–Lieb algebra does not have a diagrammatic representation.

Problem 6. Identify sufficient conditions under which a generalized Jones trace can be defined on $\mathrm{TL}(\Gamma)$, and hence on $\mathcal{H}(\Gamma)$, and use this trace to compute μ-coefficients in $\mathcal{H}(\Gamma)$.

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