# Visual Group Theory

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# Introduction

1. Welcome to VGT Summer 2009!
2. Introductions
3. Discussion of syllabus
4. Expectations and general game plan
5. My web page

   http://oz.plymouth.edu/~dcernst

6. Web page for textbook (including errata)

   http://web.bentley.edu/empl/c/ncarter/vgt/

7. Group Explorer

   http://groupexplorer.sourceforge.net/

8. OK, let's get started!

# Chapter 1: What is a group?

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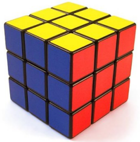
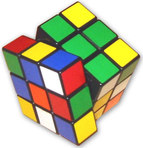
Our introduction to group theory will begin by discussing the famous Rubik's Cube.

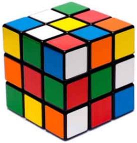
- Invented in 1974 by Ernö Rubik of Budapest, Hungary
- The cube comes out of the box in the solved position:



- But then we can scramble it up by consecutively rotating one of its 6 faces:



- The result might look something like this:



- The goal is to return the cube to its original solved position, again by consecutively rotating one of the 6 faces.

Since Rubik's Cube does not seem to require any skill with numbers to solve it, you may be inclined to think that this puzzle is not mathematical.

Group theory is not primarily about numbers, but rather about patterns and symmetry; something the Rubik's Cube possesses in abundance.

Let's explore the Rubik's Cube in more detail. In particular, let's see if we can identify some key features that will identify the boundaries of our study.

First, some questions to ponder:

- How did we scramble up the cube in the first place? How do we go about unscrambling the cube?
- In particular, what actions do we *need* in order to scramble and unscramble the cube? (There are many correct answers.)
- How is Rubik's Cube different from chess?
- How is Rubik's Cube different from poker?

4 key observations:

**Observation 1.1**

There is a predefined list of moves that never changes.

**Observation 1.2**

Every move is reversible.

**Observation 1.3**

Every move is deterministic.

**Observation 1.4**

Moves can be combined in any sequence.

We could add more to our list, but as we shall see, these 4 observations encompass the aspects of the mathematical objects that we wish to study.

Group theory studies the mathematical consequences of these 4 observations, which in turn will help us answer interesting questions about symmetrical objects.

# Rules of a group

Instead of considering our 4 observations as descriptions of Rubik's Cube, let's rephrase them as rules (axioms) that will define the boundaries of our objects of study.

Advantages of our endeavor:

1. We make it clear what it is we want to explore.
2. Helps us speak the same language, so that we know we are discussing the same objects (trapezoids. . . ).
3. The rules provide the groundwork for making logical deductions, so that we can discover new facts.

Our rules:

**Rule 1.5**

There is a predefined list of actions that never changes.

**Rule 1.6**

Every action is reversible.

**Rule 1.7**

Every action is deterministic.

**Rule 1.8**

Any sequence of consecutive actions is also an action.

What changes were made in the rephrasing?

### Comments

- We swapped the word *move* for *action*.
- The (usually short) list of actions required by Rule 1.5 is our set of building blocks; called the generators.
- Rule 1.8 tells us that any sequence of the generators is also an action.

Finally, here is our unofficial definition of a group. (We'll make things a bit more rigorous later.)

### Definition 1.9

A group is a system or collection of actions satisfying Rules 1.5–1.8.

# Group Exercises

OK, let's explore a few more examples.

1. Discuss Exercise 1.1 (see Bob = Back of book) as a large group.
2. In groups of 2–3, complete the following exercises (not collected):
   - Exercise 1.3 (see Bob)
   - Exercise 1.4
3. I'd like two groups to volunteer to discuss their answers to the two previous exercises.
4. Now, mix the groups up, so that no group stays the same. In your new groups, complete Exercise 1.8. I want each group to turn in a complete solution.

# Potential quiz questions

Here are some potential questions that I may ask you on tomorrow's quiz at the beginning of class:

1. State our unofficial definition of a group by listing the 4 rules.
2. Define generators.
3. Provide 2 examples of a group. In each case, describe a set of generators.

# References

I borrowed images from the following web pages:

- http://www.cunymath.cuny.edu/?page=mm
- http://www.math.cornell.edu/~mec/Winter2009/Lipa/Puzzles/lesson2.html